Bloch–Kato groups over perfectoid fields and Galois theory of p-adic periods

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Abstract

We relate the structure of the Bloch–Kato groups associated with a de Rham Galois representation over a perfectoid field to the Galois theory of the ring $\mathbf{B}_{\mathrm{dR}}^{+}$ of p-adic periods. As an application, we answer the question raised by Coates and Greenberg and motivated by Iwasawa theory to compute the Bloch–Kato groups over perfectoid fields in new cases, generalising results of Coates and Greenberg and the author. Our method relies on the classification of vector bundles over the Fargues–Fontaine curve.

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Contents

1 Introduction

The present article is concerned with the question raised by Coates and Greenberg [8, p. 131] and motivated by Iwasawa theory to compute the Bloch–Kato groups associated with a de Rham Galois representation over a perfectoid field. The purpose of the present article is to answer Coates and Greenberg's question in new cases, generalising the results and the method of the article [29].

1.1 Motivation

Let p be a prime number. Let $\bar{\mathbf{Q}}_p$ be an algebraic closure of the field $\mathbf{Q}_p$ of p-adic numbers. Let K be a finite extension of $\mathbf{Q}_p$ contained in $\bar{\mathbf{Q}}_p$. We denote by $G_K = \mathrm{Gal}(\bar{\mathbf{Q}}_p/K)$ the absolute Galois group of K. Let V be a p-adic representation of G_K, that is, a finite dimensional $\mathbf{Q}_p$-vector space equipped with a continuous and $\mathbf{Q}_p$-linear action of G_K, and let T be a $\mathbf{Z}_p$-lattice in V stable under the action of G_K.

Bloch and Kato [3] have defined subgroups in Galois cohomology using p-adic Hodge theory (see §6.1)

$$\mathrm{H}^1_e(K, V/T) \subseteq \mathrm{H}^1_f(K, V/T) \subseteq \mathrm{H}^1_g(K, V/T) \subseteq \mathrm{H}^1(K, V/T),$$

which are involved in their conjecture on the special values of L-functions of motives [15].

Let L be an algebraic extension of K contained in $\bar{\mathbf{Q}}_p$. For each $* \in \{e, f, g\}$, we consider the group

$$\mathrm{H}^1_*(L, V/T) = \varinjlim_{\mathrm{res}, K'} \mathrm{H}^1_*(K', V/T),$$

where K' runs over all the finite extensions of K contained in L, and the transition morphisms are the restriction maps. The Bloch–Kato subgroups thus defined

$$\mathrm{H}^1_e(L, V/T) \subseteq \mathrm{H}^1_f(L, V/T) \subseteq \mathrm{H}^1_g(L, V/T) \subseteq \mathrm{H}^1(L, V/T)$$

are involved in the Iwasawa main conjecture for motives [7, 20].

Coates and Greenberg [8, p. 131] have raised the question motivated by Iwasawa theory to compute the Bloch–Kato groups when the completion $\hat{L}$ of L for the p-adic valuation topology is a perfectoid field [30, §3]. In particular, if $\hat{L}$ is a perfectoid field, then Coates and Greenberg have computed the Bloch–Kato groups when T is the p-adic Tate module associated with an abelian variety A defined over K (see Remark 1.2.5), in which case [3, Example 3.11] V/T is the group of p-power torsion points $A[p^\infty]$ of A, and the Bloch–Kato groups are all equal and coincide with the image of the Kummer map

$$A(L) \otimes_{\mathbf{Z}} \mathbf{Q}_p/\mathbf{Z}_p \to \mathrm{H}^1(L, A[p^\infty]).$$

We refer the reader to the introduction of the articles [8, 27, 29] and the references therein for more details about this question, its history and its motivation from Iwasawa theory which can be traced back to the foundational article [26] by Mazur.

Remark 1.1.1. Coates and Greenberg [8] use the notion, which they have introduced, of deeply ramified extensions. Recall that the extension L/K is deeply ramified if and only if the field $\hat{L}$ is perfectoid (see [30, Remark 3.3] or [24, Lemma 2.21]).

Remark 1.1.2. Let $T^*(1) = \mathrm{Hom}_{\mathbf{Z}_p}(T, \mathbf{Z}_p(1))$ be the Tate dual representation of T. Let

$$\mathrm{H}^1_{\mathrm{Iw}}(K, L, T^*(1)) = \varprojlim_{\mathrm{cores}, K'} \mathrm{H}^1(K', T^*(1))$$

be the first Iwasawa cohomology group of the extension L/K with coefficients in $T^*(1)$, where K' runs over all the finite extensions of K contained in L, and the transition morphisms are the corestriction maps. For each $* \in \{e, f, g\}$, the Bloch–Kato groups are compatible under the corestriction maps and the modules of universal norms associated with $T^*(1)$ in the extension L/K are defined by

$$\mathrm{H}^1_{\mathrm{Iw},*}(K, L, T^*(1)) = \varprojlim_{\mathrm{cores}, K'} \mathrm{H}^1_*(K', T^*(1)).$$

Local Tate duality induces a perfect pairing

$$\mathrm{H}^1(L, V/T) \times \mathrm{H}^1_{\mathrm{Iw}}(K, L, T^*(1)) \to \mathbf{Q}_p/\mathbf{Z}_p.$$

If V is de Rham, then under local Tate duality (see §6.2) the groups

$$\mathrm{H}^1_e(L, V/T) \subseteq \mathrm{H}^1_f(L, V/T) \subseteq \mathrm{H}^1_g(L, V/T) \tag{1.1.1}$$

are respectively the orthogonal complements of the modules of universal norms

$$\mathrm{H}^1_{\mathrm{Iw},g}(K,L,T^*(1)) \supseteq \mathrm{H}^1_{\mathrm{Iw},f}(K,L,T^*(1)) \supseteq \mathrm{H}^1_{\mathrm{Iw},e}(K,L,T^*(1)). \qquad (1.1.2)$$

To compute the Bloch–Kato groups (1.1.1) is therefore equivalent to compute the modules of universal norms (1.1.2).

1.2 Main results

We will first prove the following relation between the Bloch–Kato groups. We consider the group $\mathrm{H}^1(L, V/T)$ and its subgroups as discrete $\mathbf{Z}_p$-modules. Recall that the Pontryagin dual of a discrete $\mathbf{Z}_p$-module M is the compact $\mathbf{Z}_p$-module $\mathrm{Hom}_{\mathbf{Z}_p}(M, \mathbf{Q}_p/\mathbf{Z}_p)$.

Proposition 1.2.1. *If V is de Rham, then the Pontryagin dual of the quotient*

$$\mathrm{H}^1_g(L, V/T)/\,\mathrm{H}^1_e(L, V/T)$$

is a free $\mathbf{Z}_p$-module of finite rank bounded independently of L.

From the perspective of Iwasawa theory, the Bloch–Kato groups are thus closely related with each other, and it is therefore enough to study one of them. We study the exponential Bloch–Kato group $\mathrm{H}^1_e(L, V/T)$.

We need the following notation to state the main result of the present article.

- The definition of the Bloch–Kato groups (see §6.1) involves the field of p-adic periods $\mathbf{B}_{\mathrm{dR}}$ and subrings of $\mathbf{B}_{\mathrm{dR}}$ introduced by Fontaine [16]. In particular, the natural filtration $(\mathrm{Fil}^n\,\mathbf{B}_{\mathrm{dR}})_{n\in\mathbf{Z}}$ on $\mathbf{B}_{\mathrm{dR}}$ induces an decreasing separated and exhaustive filtration by subgroups

  $$(\mathrm{Fil}^n\,\mathrm{H}^1_e(L, V/T))_{n\in\mathbf{Z}}$$

  on the exponential Bloch–Kato groups $\mathrm{H}^1_e(L, V/T)$.

- Let $\mathbf{B}^+_{\mathrm{dR}} = \mathrm{Fil}^0\,\mathbf{B}_{\mathrm{dR}}$ be the ring of p-adic periods, and for each integer $n \geq 1$, let $\mathbf{B}_n = \mathbf{B}^+_{\mathrm{dR}}/\,\mathrm{Fil}^n\,\mathbf{B}_{\mathrm{dR}}$. Recall that $\mathbf{B}^+_{\mathrm{dR}}$ is endowed with a canonical topology and a continuous action by G_K which induces a structure on each $\mathbf{B}_n$ of $\mathbf{Q}_p$-Banach space equipped with a continuous and $\mathbf{Q}_p$-linear action by G_K. Recall also that $\bar{\mathbf{Q}}_p$ can be identified with a subfield of $\mathbf{B}^+_{\mathrm{dR}}$, and we have $\bar{\mathbf{Q}}_p \subset \mathbf{B}_n$. In particular, there is an inclusion $L \subset \mathbf{B}_n^{G_L}$, and we consider $\mathbf{B}_n^{G_L}$ endowed with the subspace topology from $\mathbf{B}_n$.

- If V is de Rham, then, for each integer $n \geq 1$, we denote by $V^{\leq 0,>n}$ the maximal quotient representation of V whose Hodge–Tate weights are all in the set $\mathbf{Z} \setminus [1,n]$, and by $T^{\leq 0,>n}$ the image of T in $V^{\leq 0,>n}$. The quotient map $V/T \to V^{\leq 0,>n}/T^{\leq 0,>n}$ induces a morphism

  $$\pi_{0,n} : \mathrm{H}^1(L, V/T) \to \mathrm{H}^1(L, V^{\leq 0,>n}/T^{\leq 0,>n}).$$

Note that if the Hodge–Tate weights of V are all $\leq n$, then the representation $V^{\leq 0,>n}$ is the maximal quotient representation of V whose Hodge–Tate weights are all ≤ 0, and we then simply denote the representation $V^{\leq 0,>n}$ by $V^{\leq 0}$, the lattice $T^{\leq 0,>n}$ by $T^{\leq 0}$, and the map $\pi_{0,n}$ by

$$\pi_0 : \mathrm{H}^1(L, V/T) \to \mathrm{H}^1(L, V^{\leq 0}/T^{\leq 0}).$$

The main result of the present article is then the following.

Theorem 1.2.2. *Let $n \geq 1$ be an integer. If V is de Rham and if $\hat{L}$ is a perfectoid field such that L is dense in $\mathbf{B}_n^{G_L}$, then the map $\pi_{0,n}$ induces an isomorphism*

$$\mathrm{H}^1(L, V/T)/\operatorname{Fil}^{-n} \mathrm{H}^1_e(L, V/T) \xrightarrow{\sim} \mathrm{H}^1(L, V^{\leq 0,>n}/T^{\leq 0,>n}).$$

From Theorem 1.2.2, we then obtain the following descriptions of the full exponential Bloch–Kato group.

Corollary 1.2.3. *Let $n \geq 1$ be an integer. Assume that V is de Rham and that $\hat{L}$ is a perfectoid field such that L is dense in $\mathbf{B}_n^{G_L}$.*

1. *If the quotient representation $V^{\leq 0,>n}$ is trivial, then*

$$\mathrm{H}^1_e(L, V/T) = \mathrm{H}^1(L, V/T).$$

2. *If the Hodge–Tate weights of V are all $\leq n$, then the map π_0 induces an isomorphism*

$$\mathrm{H}^1(L, V/T)/\mathrm{H}^1_e(L, V/T) \xrightarrow{\sim} \mathrm{H}^1(L, V^{\leq 0}/T^{\leq 0}).$$

Theorem 1.2.2 therefore relates the structure of the Bloch–Kato groups over perfectoid fields to the Galois theory of the ring $\mathbf{B}_{\mathrm{dR}}^+$ of p-adic periods.

Example 1.2.4. We recall results about the density of L in $\mathbf{B}_n^{G_L}$.

1. Colmez [9, 11] has proved that $\bar{\mathbf{Q}}_p$ is dense in $\mathbf{B}_{\mathrm{dR}}^+$.

2. If $\hat{L}$ is *not* a perfectoid field, then Iovita and Zaharescu [22, Theorem 0.1] have proved that, for each $n \in \mathbf{N}$, there are isomorphisms

$$(\mathbf{B}_{\mathrm{dR}}^+)^{G_L} \xrightarrow{\sim} \mathbf{B}_n^{G_L} \xrightarrow{\sim} \hat{L}.$$

   Hence, if the field $\hat{L}$ *not* perfectoid, then L is dense in $(\mathbf{B}_{\mathrm{dR}}^+)^{G_L}$.

3. Let $\mathbf{C}_p$ be the completion of $\bar{\mathbf{Q}}_p$ for the p-adic valuation topology. Recall that there exists an isomorphism $\mathbf{B}_1 \xrightarrow{\sim} \mathbf{C}_p$ of p-adic Banach representations of G_K. By the Ax–Sen–Tate theorem [33], there are isomorphisms

$$\mathbf{B}_1^{G_L} \xrightarrow{\sim} \mathbf{C}_p^{G_L} \xrightarrow{\sim} \hat{L}.$$

   Hence, the field L is always dense in $\mathbf{B}_1^{G_L}$.

4. If L/K is an infinitely ramified $\mathbf{Z}_p$-extension, then the field $\hat{L}$ is perfectoid, and Berger [2] has proved that L is *not* dense in $\mathbf{B}_2^{G_L}$. The case of the cyclotomic $\mathbf{Z}_p$-extension has also been proved by Colmez [22, Proposition 8.2].

5. If $L = K(p^{1/p^\infty})$ is the extension generated over K by all the p-power roots of p, then the field $\hat{L}$ is perfectoid, and Iovita and Zaharescu [22, Corollary 8.1] have proved that L is *not* dense in $\mathbf{B}_2^{G_L}$.

6. Let $m \geq 2$. If $K = \mathbf{Q}_{p^m}$ is the unique unramified extension of $\mathbf{Q}_p$ of degree m and if $L = K^{\mathrm{ab}}$ is the maximal abelian extension of K, then the field $\hat{L}$ is perfectoid, and Iovita and Zaharescu [22, Corollary 8.2] have proved that L is dense in $\mathbf{B}_2^{G_L}$.

Iovita and Zaharescu end their article [22] with open problems concerning the Galois theory of $\mathbf{B}_{\mathrm{dR}}^+$.

Remark 1.2.5. By the Ax–Sen–Tate theorem, Theorem 1.2.2 and Corollary 1.2.3 holds for any perfectoid field $\hat{L}$ in the case $n = 1$. In particular, the point 2 of the Corollary 1.2.3 in the case $n = 1$ is the main result of the article [29] and a generalisation of the aforementioned theorem by Coates and Greenberg [8] for abelian varieties. Indeed, recall [33, 14] that the rational p-adic Tate module $V_p(A)$ associated with an abelian variety A defined over K is a de Rham p-adic representation of G_K whose Hodge–Tate weights are all in $[0, 1]$, and thus, the point 2 of the Corollary 1.2.3 applies to $V_p(A)$. The new computations of the exponential Bloch–Kato groups obtained in Theorem 1.2.2 and Corollary 1.2.3 thereby answer the question raised by Coates and Greenberg [8, p. 131] in new cases.

Remark 1.2.6. If L is the cyclotomic $\mathbf{Z}_p$-extension of K, then L is not dense in $\mathbf{B}_2^{G_L}$ by the aforementioned results of Berger and Colmez, hence Theorem 1.2.2 and Corollary 1.2.3 apply only in the case $n = 1$. However, Berger [1], generalising results by Perrin-Riou [27, 28], has computed the Bloch–Kato groups associated with a de Rham Galois representation over the cyclotomic extension without any restriction on the Hodge–Tate weights of the representation. Berger has proved that if V is de Rham and if L is the cyclotomic $\mathbf{Z}_p$-extension of K, then the map π_0 induces a surjective morphism

$$\mathrm{H}^1(L, V/T)/\,\mathrm{H}^1_e(L, V/T) \to \mathrm{H}^1(L, V^{\leq 0}/T^{\leq 0}) \to 0$$

of which the Pontryagin dual of its kernel is a finitely generated $\mathbf{Z}_p$-module.

Remark 1.2.7. Additionally to Coates and Greenberg's open question [8, p. 131] to compute the Bloch–Kato groups over perfectoid fields, Büyükboduk [6, Conjectures 2.5, 2.6, and 2.7] has conjectured that the structure of the Bloch–Kato groups over the anticyclotomic $\mathbf{Z}_p$-extension should be similar to the structure of the Bloch–Kato groups over the cyclotomic $\mathbf{Z}_p$-extension computed by Berger [1] .

Remark 1.2.8. We precise the role of perfectoid fields in Theorem 1.2.2 and Corollary 1.2.3. Let G be a connected p-divisible group of height $\mathrm{ht}(G)$ and dimension $\dim(G)$ defined over the ring of integers $\mathcal{O}_K$ of K. Let $T_p(G)$ be the p-adic Tate module of G, and let $V_p(G) = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p(G)$ be the rational p-adic Tate module of G, hence, $V_p(G)/T_p(G) = G[p^\infty]$ is the group of p-power torsion points of G. Recall [3, Example 3.10] that the exponential Bloch–Kato group $\mathrm{H}^1_e(L, G[p^\infty])$ then coincides with the image of the Kummer map

$$G(\mathcal{O}_L) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p \to \mathrm{H}^1(L, G[p^\infty]).$$

Recall [33, 14] that the representation $V_p(G)$ is de Rham with Hodge–Tate weights in $[0, 1]$ and that the quotient representation $V_p(G)^{\leq 0}$ is the rational p-adic Tate module $V_p(G^{\text{ét}})$ associated with the maximal étale quotient $G^{\text{ét}}$ of G which is therefore trivial since G is assumed to be connected. The point 2 of Corollary 1.2.3 in the case $n = 1$ then applies to G and thus the following statement holds.

- If $\hat{L}$ is a perfectoid field, then we have $\mathrm{H}^1_e(L, G[p^\infty]) = \mathrm{H}^1(L, G[p^\infty])$.

On the one hand, if $\mathrm{ht}(G) = \dim(G)$, Coates and Greenberg [8, Proposition 4.7] have proved that if L/K is an infinitely wildly ramified, then $\mathrm{H}^1_e(L, G[p^\infty]) = \mathrm{H}^1(L, G[p^\infty])$. (Coates and Greenberg state this result for abelian varieties, their proof is valid for p-divisible groups.) Note that if $\hat{L}$ is perfectoid, then L/K is infinitely wildly ramified [8, Lemma 2.12]. Thus, for specific representations, there exists a wider class of fields than perfectoid fields for which the equality $\mathrm{H}^1_e(L, V/T) = \mathrm{H}^1(L, V/T)$ holds.

On the other hand, if $\mathrm{ht}(G) > \dim(G)$, then Bondarko [4], generalising a result by Coates and Greenberg [8, Proposition 5.4], has proved that the following statements are equivalent.

1. The field $\hat{L}$ is perfectoid.
2. The valuation of L is non-discrete and $\mathrm{H}^1_e(L, G[p^\infty]) = \mathrm{H}^1(L, G[p^\infty])$.

Remark 1.2.9. Perfectoid fields are ubiquitous in Iwasawa theory. Indeed, the fields presented in Example 1.2.4 are perfectoid. Moreover, recall that if the extension L/K is Galois with Galois group a p-adic Lie group in which the inertia subgroup is infinite, then the field $\hat{L}$ is perfectoid by Sen (see [31] and [8, Theorem 2.13]). All such fields are studied in Iwasawa theory [7, 20].

Remark 1.2.10. We briefly mention applications of our results in Iwasawa theory. Results such as the main results of the present article, Theorem 1.2.2 and Corollary 1.2.3, and Berger's result [1] for the cyclotomic extension allow to precisely compare the Bloch–Kato Selmer groups to Selmer groups à la Greenberg [21]. The Bloch–Kato Selmer groups are involved in the Iwasawa main conjecture while Selmer groups à la Greenberg are more accessible to study.

1.3 Overview of the proof

The method of the present article to study the Bloch–Kato groups over perfectoid fields generalises and improves on the approach developed in [29].

The category $\mathcal{C}(G_K)$ of almost $\mathbf{C}_p$-representations of G_K is an abelian subcategory of the category of p-adic Banach representations of G_K introduced by Fontaine [18] which contains as full subcategories the category of

p-adic representations of G_K and the category of torsion $\mathbf{B}_{\mathrm{dR}}^+$-representations of G_K.

Let $n \geq 1$ be an integer. Fontaine has associated with V a short exact sequence of almost $\mathbf{C}_p$-representations of G_K

$$0 \to V \to E_+^n(V) \to \operatorname{Fil}^{-n} \hat{t}_V(\bar{\mathbf{Q}}_p) \to 0, \tag{1.3.1}$$

such that

- $\operatorname{Fil}^{-n} \hat{t}_V(\bar{\mathbf{Q}}_p)$ is a trivial $\mathbf{B}_n$-representation of G_K, that is, there exists an isomorphism in $\mathcal{C}(G_K)$

  $$\operatorname{Fil}^{-n} \hat{t}_V(\bar{\mathbf{Q}}_p) \xrightarrow{\sim} \bigoplus_{i \in [1,n]} \mathbf{B}_i^{\oplus m_i(V)},$$

  for some integers $m_i(V) \in \mathbf{N}$. Moreover, if V is de Rham, then $m_i(V)$ is the multiplicity of i as a Hodge–Tate weight of V.

- $E_+^n(V)$ is the universal extension of V by a trivial $\mathbf{B}_n$-representation in $\mathcal{C}(G_K)$.

Let $E_\delta^n(V/T)$ be the maximal discrete G_K-submodule of $E_+^n(V)/T$. Then, there is an isomorphism

$$\mathrm{H}^1(L, V/T)/\operatorname{Fil}^{-n} \mathrm{H}_e^1(L, V/T) \xrightarrow{\sim} \mathrm{H}^1(L, E_\delta^n(V/T)). \tag{1.3.2}$$

Moreover, if $\hat{L}$ is a perfectoid field such that L is dense in $\mathbf{B}_n^{G_L}$, then we will prove that the computation of $\mathrm{H}^1(L, E_\delta^n(V/T))$ reduces to the computation of $\mathrm{H}^1(L, E_+^n(V))$ which we will then carry out as follows.

Let X^{FF} be the Fargues–Fontaine curve [13], and let $\operatorname{Coh}_{X^{\mathrm{FF}}}(G_K)$ be the category of G_K-equivariant coherent sheaves over X^{FF}. Fontaine [19] has proved that the global section functor on X^{FF} induces an equivalence of triangulated categories between the bounded derived categories

$$\mathrm{D}^b(\operatorname{Coh}_{X^{\mathrm{FF}}}(G_K)) \xrightarrow{\sim} \mathrm{D}^b(\mathcal{C}(G_K)). \tag{1.3.3}$$

Moreover, if $\hat{L}$ is a perfectoid field, then Fargues and Fontaine have classified the G_L-equivariant coherent sheaves over X^{FF}. We will deduce from these results the following.

Proposition 1.3.1. *Let X be an almost $\mathbf{C}_p$-representation of G_K. Let X^0 be the maximal quotient p-adic representation of X. If $\hat{L}$ is a perfectoid field, then the quotient map $X \to X^0$ induces an isomorphism*

$$\mathrm{H}^1(L, X) \xrightarrow{\sim} \mathrm{H}^1(L, X^0).$$

Therefore, in order to compute $\mathrm{H}^1(L, E_+^n(V))$, it remains to identify the maximal quotient p-adic representation $E_+^n(V)^0$ of $E_+^n(V)$ which we will do by considering $E_+^n(V)$ in terms of vector bundles over the Fargues–Fontaine curve.

By the equivalence (1.3.3), the short exact sequence (1.3.1) is isomorphic to the global sections of a short exact sequence of G_K-equivariant coherent sheaves over X^{FF}

$$0 \to \mathcal{E}(V) \xrightarrow{\eta} \mathcal{E}_+^n(V) \to \mathcal{F}_+^n(V) \to 0,$$

where $\mathcal{E}(V) = \mathcal{O}_{X^{\mathrm{FF}}} \otimes_{\mathbf{Q}_p} V$ is the vector bundle associated with V over X^{FF}. If V is de Rham, then $\mathcal{E}(V)$ and $\mathcal{E}^n_+(V)$ are de Rham vector bundles over X^{FF}, and we will prove that $\eta : \mathcal{E}(V) \to \mathcal{E}^n_+(V)$ is then solution of the following universal problem.

Let $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}$ be the category of de Rham G_K-equivariant vector bundles over X^{FF}, and let $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)^{\leq 0,>n}_{\mathrm{dR}}$ be the full subcategory of $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}$ of de Rham G_K-equivariant vector bundles over X^{FF} whose Hodge–Tate weights are all in the set $\mathbf{Z}\backslash[1,n]$. The forgetful functor from $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)^{\leq 0,>n}_{\mathrm{dR}}$ to $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}$ admits a left adjoint

$$\begin{aligned} \tau^{\leq 0,>n}_{\mathrm{dR}} : \mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}} &\to \mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)^{\leq 0,>n}_{\mathrm{dR}} \\ \mathcal{E} &\mapsto \mathcal{E}^n_+. \end{aligned}$$

If V is de Rham, then $\mathcal{E}^n_+(V) = \tau^{\leq 0,>n}_{\mathrm{dR}}(\mathcal{E}(V))$ and $\eta : \mathcal{E}(V) \to \mathcal{E}^n_+(V)$ is the universal morphism from $\mathcal{E}(V)$ to the forgetful functor from $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)^{\leq 0,>n}_{\mathrm{dR}}$ to $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}$. We will then easily deduce from the universal property of $\mathcal{E}^n_+(V)$ that the maximal quotient p-adic representation $E^n_+(V)^0$ of $E^n_+(V)$ is the representation $V^{\leq 0,>n}$.

1.4 Organisation of the article

In section 2, we review the definition and some properties of almost $\mathbf{C}_p$-representations due to Fontaine [18, 19]. We also briefly review the p-adic period rings defined by Fontaine [16].

In section 3, we review the properties of coherent sheaves over the Fargues–Fontaine curve [13] which will be needed. We recall the relation (1.3.3) between almost $\mathbf{C}_p$-representations and coherent sheaves over the Fargues–Fontaine curve established by Fontaine [19]. We study the maximal quotient p-adic representation of an almost $\mathbf{C}_p$-representation.

In section 4, we establish Proposition 1.3.1. Then, given an almost $\mathbf{C}_p$-representation X, an almost $\mathbf{C}_p$-subrepresentation Z of X, and a Galois stable lattice $\mathcal{Z}$ in Z, we study the cohomology of the maximal discrete Galois submodule $(X/\mathcal{Z})_\delta$ of $X/\mathcal{Z}$, which will allow us to relate the cohomology of $E^n_\delta(V/T)$ to the cohomology of $E^n_+(V)$.

In section 5, we define and study the functor $\tau^{\leq 0,>n}_{\mathrm{dR}}$. We then study the vector bundle $\mathcal{E}^n_+(V)$.

In section 6, we review the definition of the Bloch–Kato groups and we define the filtration on the exponential Bloch–Kato group. We then review the definition and properties of the almost $\mathbf{C}_p$-representation $E^n_+(V)$, and of the discrete Galois module $E^n_\delta(V/T)$ associated with V/T by Fontaine. We then establish the relation (1.3.2) between the cohomology of $E^n_\delta(V/T)$ and the filtered part of the exponential Bloch–Kato group, and the relation between $E^n_+(V)$ and $\mathcal{E}^n_+(V)$. Finally, we prove the main results stated in the introduction: Proposition 1.2.1, Theorem 1.2.2, and Corollary 1.2.3.

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1.6 Notation

We adopt the convention that the set of natural numbers $\mathbf{N}$ contains 0.

We fix a prime number p, and an algebraic closure $\bar{\mathbf{Q}}_p$ of the field $\mathbf{Q}_p$ of p-adic numbers. We denote by $\mathbf{C}_p$ the completion of $\bar{\mathbf{Q}}_p$ for the p-adic valuation topology. Every algebraic extension of $\mathbf{Q}_p$ considered is contained in $\bar{\mathbf{Q}}_p$. We denote by $\mathbf{Q}_p^{\mathrm{ur}}$ the maximal unramified extension of $\mathbf{Q}_p$. If L is an algebraic extension of $\mathbf{Q}_p$, then we denote by $G_L = \mathrm{Gal}(\bar{\mathbf{Q}}_p/L)$ the absolute Galois group of L, by $L_0 = L \cap \mathbf{Q}_p^{\mathrm{ur}}$ the maximal unramified extension of $\mathbf{Q}_p$ contained in L, and by $\hat{L}$ the completion of L for the p-adic valuation topology. We also fix a finite extension K of $\mathbf{Q}_p$.

If G is a topological group, then a topological G-module is a topological abelian group M equipped with a continuous action of G compatible with the group structure of M, and a discrete G-module is a topological G-module whose underlying topological space is discrete. If G is a topological group, and if M is a topological G-module, then, for each $n \in \mathbf{N}$, we denote by $\mathrm{H}^n(G, M)$ the n-th group of continuous group cohomology of G with coefficients in M (see [34, §2] or [29, Appendix A]). Recall that if

$$0 \to M' \to M \to M'' \to 0 \tag{1.6.1}$$

is a short exact sequence of topological G-modules such that the topology of M' is the subspace topology from M, and the topology of M'' is the quotient topology from M, then the short exact sequence (1.6.1) induces an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^0(G, M') \arrow[r] & \mathrm{H}^0(G, M) \arrow[r] & \mathrm{H}^0(G, M'') \arrow[dll, rounded corners] \\ & \mathrm{H}^1(G, M') \arrow[r] & \mathrm{H}^1(G, M) \arrow[r] & \mathrm{H}^1(G, M''). \end{tikzcd} \tag{1.6.2}$$

Moreover, if there exists a continuous section of the projection of M on M'' as topological space, then the exact sequence (1.6.2) extends into a long exact sequence

$$\cdots \to \mathrm{H}^n(G,M') \to \mathrm{H}^n(G,M) \to \mathrm{H}^n(G,M'') \to \mathrm{H}^{n+1}(G,M') \to \cdots.$$

In particular, such a continuous section exists if M'' is a discrete G-module.

If G_k is the absolute Galois group of a field k, and if M is a topological G_k-module, then, for each $n \in \mathbf{N}$, we write $\mathrm{H}^n(k,M)$ instead of $\mathrm{H}^n(G_k,M)$, and $\mathrm{H}^n(k,M)$ is the n-th group of Galois cohomology of k with coefficients in M.

We denote by $\mathbf{Z}_p(1)$ the free $\mathbf{Z}_p$-module of rank 1 whose elements are sequences $(\zeta_{p^n})_{n\in\mathbf{N}}$ of p-power roots of unity in $\bar{\mathbf{Q}}_p$ such that $\zeta_1 = 1$ and $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for each $n \in \mathbf{N}$, endowed with the natural action of $G_{\mathbf{Q}_p}$ by the cyclotomic character χ. We fix a generator t of $\mathbf{Z}_p(1)$ with group law written additively. For each $n \in \mathbf{N}$, we set

$$\begin{aligned}\mathbf{Z}_p(n) &= \mathrm{Sym}^n_{\mathbf{Z}_p}(\mathbf{Z}_p(1)),\\ \mathbf{Z}_p(-n) &= \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{Z}_p(n),\mathbf{Z}_p).\end{aligned}$$

Then, for each $n \in \mathbf{Z}$, the Galois group $G_{\mathbf{Q}_p}$ acts on $\mathbf{Z}_p(n)$ by χ^n.

If L is an algebraic extension of $\mathbf{Q}_p$ and if M is a $\mathbf{Z}_p$-module equipped with a linear action of G_L, then, for each $n \in \mathbf{Z}$, then the *n-th Tate twist* $M(n)$ of M is the $\mathbf{Z}_p$-module

$$M(n) = M \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(n)$$

on which G_L acts by $g(m \otimes z) = g(m) \otimes g(z) = \chi^n(g)(g(m) \otimes z)$, for all $g \in G_L$, $m \in M$ and $z \in \mathbf{Z}_p(n)$.

2 Almost $\mathbf{C}_p$-representations

2.1 p-adic Banach representations and almost $\mathbf{C}_p$-representations

We recall the definition of the category of almost $\mathbf{C}_p$-representations of G_K introduced by Fontaine [18].

A *p-adic Banach representation of G_K* is a $\mathbf{Q}_p$-Banach space equipped with a continuous and $\mathbf{Q}_p$-linear action of G_K. A morphism of p-adic Banach representations of G_K is a G_K-equivariant continuous and $\mathbf{Q}_p$-linear map. We denote by $\mathcal{B}(G_K)$ the category of p-adic Banach representations of G_K.

A *G_K-stable lattice* in a p-adic Banach representation X of G_K is a $\mathbf{Z}_p$-submodule $\mathcal{X}$ of X which is complete and separated for the p-adic topology, stable under the action of G_K and such that the inclusion of $\mathcal{X}$ in X induces an isomorphism $\mathcal{X} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \xrightarrow{\sim} X$.

Note that the field $\mathbf{C}_p$ endowed with its natural topology and its natural action of G_K is a p-adic Banach representation of G_K. A *trivial $\mathbf{C}_p$-representation of G_K* is a p-adic Banach representation of G_K isomorphic to $\mathbf{C}_p^{\oplus d}$ for some $d \in \mathbf{N}$.

If X and Y are two p-adic Banach representations of G_K, then an *almost isomorphism* from X to Y is a triple (V, U, α) composed of:

1. a finite dimensional G_K-stable $\mathbf{Q}_p$-vector subspace V of X,
2. a finite dimensional G_K-stable $\mathbf{Q}_p$-vector subspace U of Y, and
3. an isomorphism in $\mathcal{B}(G_K)$

$$\alpha : X/V \xrightarrow{\sim} Y/U.$$

Almost isomorphisms form an equivalence relation on p-adic Banach representations of G_K. Two p-adic Banach representations of G_K are almost isomorphic if there exists an almost isomorphism between them.

An *almost* $\mathbf{C}_p$*-representation of* G_K is a p-adic Banach representation of G_K which is almost isomorphic to a trivial $\mathbf{C}_p$-representation of G_K. We denote by $\mathcal{C}(G_K)$ the full subcategory of $\mathcal{B}(G_K)$ of almost $\mathbf{C}_p$-representations of G_K.

Theorem 2.1.1 (Fontaine). *The category $\mathcal{C}(G_K)$ is abelian.*

Let $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ be the category of *p-adic representations of* G_K, that is, the category of finite dimensional $\mathbf{Q}_p$-vector spaces equipped with a continuous and $\mathbf{Q}_p$-linear action of G_K. The category $\mathrm{Rep}_{\mathbf{Q}_p}(G_K)$ defines a Serre subcategory of $\mathcal{C}(G_K)$ which we denote by $\mathcal{C}^0(G_K)$.

2.2 p-adic period rings

We briefly review the p-adic period rings defined by Fontaine [16].

The *ring of p-adic periods* $\mathbf{B}_{\mathrm{dR}}^+$ is a complete discrete valuation ring endowed with an action of $G_{\mathbf{Q}_p}$, whose residue field is $\mathbf{C}_p$, and of which t is a uniformiser.

The *field of p-adic periods* $\mathbf{B}_{\mathrm{dR}}$ is the field of fractions of $\mathbf{B}_{\mathrm{dR}}^+$. There is a natural filtration on $\mathbf{B}_{\mathrm{dR}}$ by the fractional ideals

$$\mathrm{Fil}^n \mathbf{B}_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}}^+ \cdot t^n, n \in \mathbf{Z},$$

which is stable under the action of $G_{\mathbf{Q}_p}$. For each $n \in \mathbf{N}$, we set

$$\mathbf{B}_n = \mathbf{B}_{\mathrm{dR}}^+ / \mathrm{Fil}^n \mathbf{B}_{\mathrm{dR}}.$$

In particular, there is an isomorphism $\mathbf{B}_1 \xrightarrow{\sim} \mathbf{C}_p$.

The field $\mathbf{B}_{\mathrm{dR}}$ is equipped with a topology, the so-called *canonical* topology (see [16, §1.5.3] or [11]), which is coarser than the valuation topology from $\mathbf{B}_{\mathrm{dR}}^+$. The action of $G_{\mathbf{Q}_p}$ on $\mathbf{B}_{\mathrm{dR}}$ endowed with the canonical topology is continuous, and, we have

$$\begin{aligned} (\mathrm{Fil}^n \mathbf{B}_{\mathrm{dR}})^{G_K} &= \mathbf{B}_{\mathrm{dR}}^{G_K} = K \quad && \text{if } n \le 0, \\ (\mathrm{Fil}^n \mathbf{B}_{\mathrm{dR}})^{G_K} &= 0 && \text{if } n > 0. \end{aligned} \tag{2.2.1}$$

Unless otherwise stated, we will consider $\mathbf{B}_{\mathrm{dR}}$ and its subquotient rings endowed with the canonical topology. In particular, for each integer $n > 0$,

$\mathbf{B}_n$ is a p-adic Banach representation of $G_{\mathbf{Q}_p}$, and the isomorphism $\mathbf{B}_1 \xrightarrow{\sim} \mathbf{C}_p$ is an isomorphism of p-adic Banach representations of $G_{\mathbf{Q}_p}$.

Moreover, the map $\mathbf{B}_{\mathrm{dR}}^+ \to \mathbf{C}_p$ induces an isomorphism between the separable closure of $\mathbf{Q}_p$ in $\mathbf{B}_{\mathrm{dR}}^+$ and the field of the p-adic algebraic numbers $\bar{\mathbf{Q}}_p$, which we use to identify these two fields. The field $\bar{\mathbf{Q}}_p$ is then dense in $\mathbf{B}_{\mathrm{dR}}^+$ ([9, 11]).

The ring $\mathbf{B}_{\mathrm{dR}}^+$ contains a subring $\mathbf{B}_{\mathrm{cris}}^+$, stable under the action of $G_{\mathbf{Q}_p}$, containing t, and equipped with an endomorphism φ commuting with the action of $G_{\mathbf{Q}_p}$. The *ring of crystalline periods* is $\mathbf{B}_{\mathrm{cris}} = \mathbf{B}_{\mathrm{cris}}^+[1/t]$. We have

$$\mathbf{B}_{\mathrm{cris}}^{G_K} = K_0.$$

Moreover, we have $\varphi(t) = pt$, and the endomorphism φ extends uniquely to $\mathbf{B}_{\mathrm{cris}}$.

Let

$$\mathbf{B}_{\mathrm{e}} = \mathbf{B}_{\mathrm{cris}}^{\varphi=1} = \{b \in \mathbf{B}_{\mathrm{cris}}, \varphi(b) = b\},$$

and, for each $n \in \mathbf{N}$, let

$$(\mathbf{B}_{\mathrm{cris}}^+)^{\varphi=p^n} = \{b \in \mathbf{B}_{\mathrm{cris}}^+, \varphi(b) = p^n b\}.$$

The ring $\mathbf{B}_{\mathrm{e}}$ is a principal ideal domain [13, Théorème 6.5.2]. Moreover, the ring $\mathbf{B}_{\mathrm{e}}$ inherits a filtration from $\mathbf{B}_{\mathrm{dR}}$, and we have

$$\mathrm{Fil}^n \mathbf{B}_{\mathrm{e}} = \mathrm{Fil}^n \mathbf{B}_{\mathrm{dR}} \cap \mathbf{B}_{\mathrm{e}} = \begin{cases} (\mathbf{B}_{\mathrm{cris}}^+)^{\varphi=p^{-n}} \cdot t^n & \text{if } n \leq 0, \\ 0 & \text{if } n > 0. \end{cases}$$

Furthermore, we have

$$\mathrm{Fil}^0 \mathbf{B}_{\mathrm{e}} = (\mathbf{B}_{\mathrm{cris}}^+)^{\varphi=1} = \mathbf{Q}_p,$$

and there exists $G_{\mathbf{Q}_p}$-equivariant short exact sequences of topological $\mathbf{Q}_p$-algebras, the so-called *fundamental exact sequences*,

$$0 \to \mathbf{Q}_p \to \mathbf{B}_{\mathrm{e}} \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^+ \to 0, \tag{2.2.2}$$

and, for each $n \in \mathbf{N}$,

$$0 \to \mathbf{Q}_p \to \mathrm{Fil}^{-n} \mathbf{B}_{\mathrm{e}} \to \mathrm{Fil}^{-n} \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^+ \to 0. \tag{2.2.3}$$

2.3 Almost $\mathbf{C}_p$-representations and p-adic periods

We recall the relation between $\mathbf{B}_{\mathrm{dR}}^+$-representations of G_K and almost $\mathbf{C}_p$-representations of G_K established by Fontaine [18].

Let $\mathrm{Rep}_{\mathbf{B}_{\mathrm{dR}}^+}^{\mathrm{tor}}(G_K)$ be the category of finitely generated torsion $\mathbf{B}_{\mathrm{dR}}^+$-modules equipped with a continuous and $\mathbf{B}_{\mathrm{dR}}^+$-semilinear action of G_K. We call an object of $\mathrm{Rep}_{\mathbf{B}_{\mathrm{dR}}^+}^{\mathrm{tor}}(G_K)$ a *torsion* $\mathbf{B}_{\mathrm{dR}}^+$*-representation of* G_K.

Note that the $\mathbf{B}_{\mathrm{dR}}^+$-module underlying a torsion $\mathbf{B}_{\mathrm{dR}}^+$-representation of G_K is a $\mathbf{B}_n$-module for some sufficiently large integer n. Since $\mathbf{B}_n$ is a $\mathbf{Q}_p$-Banach space, a torsion $\mathbf{B}_{\mathrm{dR}}^+$-representation of G_K is a p-adic Banach representation of G_K. Hence, there is a forgetful functor from $\mathrm{Rep}_{\mathbf{B}_{\mathrm{dR}}^+}^{\mathrm{tor}}(G_K)$ to $\mathcal{B}(G_K)$ whose essential image we denote by $\mathcal{C}^{+\infty}(G_K)$.

Theorem 2.3.1 (Fontaine)**.** *The forgetful functor from* $\mathrm{Rep}^{\mathrm{tor}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K)$ *to* $\mathcal{B}(G_K)$ *is fully faithful. Moreover, its essential image* $\mathcal{C}^{+\infty}(G_K)$ *is a weak Serre subcategory of* $\mathcal{C}(G_K)$*.*

2.4 Torsion pairs

We recall the definition and properties of torsion pairs on a abelian category [5, §1.12].

Definition 2.4.1. Let $\mathcal{A}$ be an abelian category. A *torsion pair* on $\mathcal{A}$ is a tuple $(\mathcal{B}, \mathcal{C})$ of strictly full subcategories of $\mathcal{A}$ such that

1. for each object B of $\mathcal{B}$ and each object C of $\mathcal{C}$, we have
$$\mathrm{Hom}_{\mathcal{A}}(B, C) = 0,$$
2. for each object A of $\mathcal{A}$, there exists a short exact sequence
$$0 \to B \to A \to C \to 0,$$
with B an object of $\mathcal{B}$, and C an object of $\mathcal{C}$.

If $(\mathcal{B}, \mathcal{C})$ is a torsion pair on an abelian category $\mathcal{A}$, then the definition of a torsion pair implies that for each object A of $\mathcal{A}$, there exists a unique, up to isomorphism, short exact sequence
$$0 \to B \to A \to C \to 0,$$
with B an object of $\mathcal{B}$, and C an object of $\mathcal{C}$ (see [5, Proposition 1.12.4]).

2.5 Effective and coeffective almost $\mathbf{C}_p$-representations

We recall the definition of effective and coeffective almost $\mathbf{C}_p$-representations of G_K due to Fontaine [19, §3L, §6C].

An almost $\mathbf{C}_p$-representation X of G_K is *effective* if there exists an object Y of $\mathcal{C}^{+\infty}(G_K)$ such that X is isomorphic to a subobject of Y. We denote by $\mathcal{C}^{\geq 0}(G_K)$ the full subcategory of $\mathcal{C}(G_K)$ of effective almost $\mathbf{C}_p$-representations of G_K.

An almost $\mathbf{C}_p$-representation X of G_K is *coeffective* if for all object Y of $\mathcal{C}^{+\infty}(G_K)$, we have
$$\mathrm{Hom}_{\mathcal{C}(G_K)}(X, Y) = 0.$$
We denote by $\mathcal{C}^{<0}(G_K)$ the full subcategory of $\mathcal{C}(G_K)$ of coeffective almost $\mathbf{C}_p$-representations of G_K.

Proposition 2.5.1 (Fontaine)**.** *The subcategories* $\mathcal{C}^{<0}(G_K)$ *and* $\mathcal{C}^{\geq 0}(G_K)$ *of* $\mathcal{C}(G_K)$ *satisfy the following properties.*

1. *The categories* $\mathcal{C}^{<0}(G_K)$ *and* $\mathcal{C}^{\geq 0}(G_K)$ *are exact subcategories of* $\mathcal{C}(G_K)$*.*
2. *The category* $\mathcal{C}^{\geq 0}(G_K)$ *is the smallest strictly full subcategory of* $\mathcal{C}(G_K)$ *containing* $\mathcal{C}^{0}(G_K)$ *and* $\mathcal{C}^{+\infty}(G_K)$ *and stable under taking extensions and direct summands.*
3. *The tuple* $(\mathcal{C}^{<0}(G_K), \mathcal{C}^{\geq 0}(G_K))$ *is a torsion pair on* $\mathcal{C}(G_K)$*.*

3 Coherent sheaves over the Fargues–Fontaine curve

3.1 The Fargues–Fontaine curve

We review properties of coherent sheaves over the Fargues–Fontaine curve [13].

Let

$$X^{\mathrm{FF}} = \mathrm{Proj}\left(\bigoplus_{n\in\mathbf{N}} (\mathbf{B}^+_{\mathrm{cris}})^{\varphi=p^n}\right)$$

be the Fargues–Fontaine curve. Recall that the scheme X^{FF} is regular, Noetherian, separated, connected, and one-dimensional. Moreover, the curve X^{FF} is complete.

We recall the description à la Beauville–Laszlo of coherent sheaves over X^{FF}.

Let L be an algebraic extension of $\mathbf{Q}_p$. For a ring $R \in \{\mathbf{B}_{\mathrm{e}}, \mathbf{B}^+_{\mathrm{dR}}, \mathbf{B}_{\mathrm{dR}}\}$, we denote by $\mathrm{Rep}_R(G_L)$ the category of finitely generated R-modules equipped with a continuous and R-semilinear action of G_L, and an object of $\mathrm{Rep}_R(G_L)$ is called a R-representation of G_L.

Let $\mathcal{M}(G_L)$ be the category whose objects are triple $(M_{\mathrm{e}}, M^+_{\mathrm{dR}}, \iota_M)$ composed of:

1. a $\mathbf{B}_{\mathrm{e}}$-representation M_{e} of G_L,
2. a $\mathbf{B}^+_{\mathrm{dR}}$-representation M^+_{dR} of G_L, and
3. a G_L-equivariant morphism of $\mathbf{B}^+_{\mathrm{dR}}$-modules

   $$\iota_M : M^+_{\mathrm{dR}} \to \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{e}}} M_{\mathrm{e}}$$

   which induces an isomorphism of $\mathbf{B}_{\mathrm{dR}}$-representations of G_L

   $$\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}^+_{\mathrm{dR}}} M^+_{\mathrm{dR}} \xrightarrow{\sim} \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{e}}} M_{\mathrm{e}},$$

and whose maps are tuple $f = (f_{\mathrm{e}}, f^+_{\mathrm{dR}})$ composed of:

1. a morphism f_{e} of $\mathbf{B}_{\mathrm{e}}$-representations of G_L, and
2. a morphism f^+_{dR} of $\mathbf{B}^+_{\mathrm{dR}}$-representations of G_L,

such that the diagram

$$\begin{tikzcd}
M^+_{\mathrm{dR}} \arrow[r, "\iota_M"] \arrow[d, "f^+_{\mathrm{dR}}"] & \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{e}}} M_{\mathrm{e}} \arrow[d, "1\otimes f_{\mathrm{e}}"] \\
N^+_{\mathrm{dR}} \arrow[r, "\iota_N"] & \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{e}}} N_{\mathrm{e}}
\end{tikzcd}$$

commutes.

The Galois group $G_{\mathbf{Q}_p}$ acts on the Fargues–Fontaine curve X^{FF} via its action on the period rings defining X^{FF}. We denote by $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_L)$ the category of G_L-equivariant coherent sheaves over X^{FF}.

Recall that there exists a unique closed point of X^{FF}, denoted by ∞, fixed by the action of $G_{\mathbf{Q}_p}$ on X^{FF}. The completion of the stalk $\mathcal{O}_{X^{\mathrm{FF}},\infty}$ is isomorphic to $\mathbf{B}_{\mathrm{dR}}^+$, and $X^{\mathrm{FF}} \setminus \{\infty\} = \mathrm{Spec}(\mathbf{B}_{\mathrm{e}})$. Therefore, there is a functor

$$\mathrm{Coh}_{X^{\mathrm{FF}}}(G_L) \to \mathcal{M}(G_L) \tag{3.1.1}$$

which associates with a coherent sheaf $\mathcal{F}$ the triple:

- $\mathcal{F}_{\mathrm{e}} = \mathrm{H}^0(X^{\mathrm{FF}} \setminus \{\infty\}, \mathcal{F})$,
- $\mathcal{F}_{\mathrm{dR}}^+$ the completion of the stalk $\mathcal{F}_\infty$, and
- $\iota_{\mathcal{F}}$ the glueing data.

Proposition 3.1.1 (Fargues–Fontaine)**.** *The functor* (3.1.1) *induces an equivalence of categories*

$$\mathrm{Coh}_{X^{\mathrm{FF}}}(G_L) \xrightarrow{\sim} \mathcal{M}(G_L).$$

We denote by $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_L)$ the full subcategory of $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_L)$ of G_L-equivariant vector bundles over X^{FF}.

Under the equivalence of Theorem 3.1.1, the category $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_L)$ is equivalent to the full subcategory of $\mathcal{M}(G_L)$ whose objects are triple $(M_{\mathrm{e}}, M_{\mathrm{dR}}^+, \iota_M)$ such that

- the $\mathbf{B}_{\mathrm{e}}$-module underlying M_{e} is a free, and
- the $\mathbf{B}_{\mathrm{dR}}^+$-module underlying M_{dR}^+ is a free.

Moreover, if L is a finite extension of $\mathbf{Q}_p$, then Fontaine [19, Proposition 3.1] has proved that the $\mathbf{B}_{\mathrm{e}}$-module underlying any $\mathbf{B}_{\mathrm{e}}$-representation of G_L is free. Therefore, we have the following characterisation of G_K-equivariant vector bundles and G_K-equivariant torsion coherent sheaves over X^{FF}.

Proposition 3.1.2. *Let $\mathcal{F}$ be G_K-equivariant coherent sheaf over X^{FF}.*

1. *The sheaf $\mathcal{F}$ is a vector bundle if and only if the $\mathbf{B}_{\mathrm{dR}}^+$-module underlying $\mathcal{F}_{\mathrm{dR}}^+$ is free.*
2. *The following statements are equivalent.*
   (a) *The sheaf $\mathcal{F}$ is torsion.*
   (b) *The $\mathbf{B}_{\mathrm{e}}$-representation $\mathcal{F}_{\mathrm{e}}$ is trivial.*
   (c) *The $\mathbf{B}_{\mathrm{dR}}^+$-representation $\mathcal{F}_{\mathrm{dR}}^+$ is torsion.*

Notation 3.1.3. We will write $\mathcal{F} = (\mathcal{F}_{\mathrm{e}}, \mathcal{F}_{\mathrm{dR}}^+, \iota_{\mathcal{F}})$ a coherent sheaf over X^{FF}. We will omit the map $\iota_{\mathcal{F}}$ when there is no ambiguity.

Notation 3.1.4. If M_{dR}^+ denotes a $\mathbf{B}_{\mathrm{dR}}^+$-module, then we set

$$M_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{dR}}^+} M_{\mathrm{dR}}^+.$$

3.2 The Harder–Narasimhan filtration

We denote by $\mathrm{K}_0(\mathrm{Coh}_{X^{\mathrm{FF}}})$ the Grothendieck group associated with the category $\mathrm{Coh}_{X^{\mathrm{FF}}}$ of coherent sheaves over X^{FF}. There exists group homomorphisms *degree* and *rank*

$$\begin{aligned} \deg &: \mathrm{K}_0(\mathrm{Coh}_{X^{\mathrm{FF}}}) \to \mathbf{Z}, \\ \mathrm{rank} &: \mathrm{K}_0(\mathrm{Coh}_{X^{\mathrm{FF}}}) \to \mathbf{Z}, \end{aligned}$$

characterised by the following properties.

- If $\mathcal{F}$ is a coherent sheaf over X^{FF}, then $\mathrm{rank}(\mathcal{F})$ is the rank of $\mathcal{F}$ as an $\mathcal{O}_{X^{\mathrm{FF}}}$-module.
- If $\mathcal{L}$ is an invertible sheaf over X^{FF}, then $\deg(\mathcal{L})$ is the degree of the divisor associated with $\mathcal{L}$.
- If $\mathcal{E}$ is a vector bundle over X^{FF} of rank r, and if $\bigwedge^r \mathcal{E}$ is the determinant line bundle associated with $\mathcal{E}$, then
$$\deg(\mathcal{E}) = \deg\left(\bigwedge^r \mathcal{E}\right).$$
- If $\mathcal{F}$ is a torsion coherent sheaf over X^{FF}, then
$$\deg(\mathcal{F}) = \sum_{\text{closed point } x \in X^{\mathrm{FF}}} \mathrm{length}_{\mathcal{O}_{X^{\mathrm{FF}},x}}(\mathcal{F}_x).$$

The *slope* of a coherent sheaf $\mathcal{F}$, denoted by $\mu(\mathcal{F})$, is the element of $\mathbf{Q} \cup \{+\infty\}$ defined by

$$\mu(\mathcal{F}) = \begin{cases} +\infty & \text{if } \mathcal{F} \text{ is torsion,} \\ \dfrac{\deg(\mathcal{F})}{\mathrm{rank}(\mathcal{F})} & \text{otherwise.} \end{cases}$$

A coherent sheaf $\mathcal{F}$ is *semistable* if for each non-zero subsheaf $\mathcal{F}' \subseteq \mathcal{F}$, we have $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$.

If $\mathcal{F}$ is a coherent sheaf over X^{FF}, there exists a unique filtration of $\mathcal{F}$ by subsheaves

$$0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_{n-1} \subseteq \mathcal{F}_n = \mathcal{F} \tag{3.2.1}$$

such that

1. the sheaf $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semistable for each $i \in \{1, \dots, n\}$, and
2. $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) > \mu(\mathcal{F}_{i+1}/\mathcal{F}_i)$ for each $i \in \{1, \dots, n-1\}$.

The filtration (3.2.1) is the *Harder–Narasimhan filtration* of $\mathcal{F}$, and the slopes $(\mu(\mathcal{F}_i/\mathcal{F}_{i-1}))_{i \in \{1,\dots,n\}}$ are the *Harder–Narasimhan slopes* of $\mathcal{F}$.

Proposition 3.2.1 (Fargues–Fontaine). *Let $\mathcal{F}$ be a coherent sheaf over X^{FF}. For each integer $n > 1$, the group $\mathrm{H}^n(X^{\mathrm{FF}}, \mathcal{F})$ is trivial. There is an exact sequence functorial in $\mathcal{F}$*

$$0 \to \mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{F}) \to \mathcal{F}_{\mathrm{e}} \oplus \mathcal{F}_{\mathrm{dR}}^+ \xrightarrow{\delta_{\mathcal{F}}} \mathcal{F}_{\mathrm{dR}} \to \mathrm{H}^1(X^{\mathrm{FF}}, \mathcal{F}) \to 0,$$

where $\delta_{\mathcal{F}}(x, y) = x - \iota_{\mathcal{F}}(y)$. Moreover,

1. *the group* $\mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{F})$ *is trivial if and only if the Harder–Narasimhan slopes of* $\mathcal{F}$ *are all* < 0*, and*

2. *the group* $\mathrm{H}^1(X^{\mathrm{FF}}, \mathcal{F})$ *is trivial if and only if the Harder–Narasimhan slopes of* $\mathcal{F}$ *are all* ≥ 0*.*

Let L be an algebraic extension of $\mathbf{Q}_p$.

Remark 3.2.2. The uniqueness of the Harder–Narasimhan filtration implies that the Harder–Narasimhan filtration of a G_L-equivariant coherent sheaf is composed of G_L-equivariant coherent sheaves.

If $\hat{L}$ is a perfectoid field [30, §3], then Fargues and Fontaine [13, Théorème 9.3.1 and Théorème 9.4.1] have classified G_L-equivariant sheaves over X^{FF}. Part of the classification is the following.

Theorem 3.2.3 (Fargues–Fontaine)**.** *If* $\hat{L}$ *is a perfectoid field, then the Harder–Narasimhan filtration of a* G_L*-equivariant coherent sheaf over* X^{FF} *is split in* $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_L)$*.*

3.3 Harder–Narasimhan twists

We recall the definition and properties of the Harder–Narasimhan twists of coherent sheaves due to Fontaine [19, §3H]. If $\mathcal{F}$ is a coherent sheaf over X^{FF}, then, for each $n \in \mathbf{Z}$, the *n-th Harder–Narasimhan twist* of $\mathcal{F}$, denoted by $\mathcal{F}(n)_{\mathrm{HN}}$, is the coherent sheaf defined as the following modification of $\mathcal{F}$ at the point ∞:

$$\mathcal{F}(n)_{\mathrm{HN}} = (\mathcal{F}_{\mathrm{e}}, \mathcal{F}^+_{\mathrm{dR}}(-n), \iota_{\mathcal{F}}(-n)).$$

We then have the following short exact sequences of coherent sheaves

$$0 \to \mathcal{F} \to \mathcal{F}(n)_{\mathrm{HN}} \to (0, t^{-n}\mathcal{F}^+_{\mathrm{dR}}/\mathcal{F}^+_{\mathrm{dR}}) \to 0 \quad \text{if } n \geq 0, \tag{3.3.1}$$

$$0 \to \mathcal{F}(n)_{\mathrm{HN}} \to \mathcal{F} \to (0, \mathcal{F}^+_{\mathrm{dR}}/t^{-n}\mathcal{F}^+_{\mathrm{dR}}) \to 0 \quad \text{if } n < 0. \tag{3.3.2}$$

Moreover, if L is an algebraic extension of $\mathbf{Q}_p$ and if $\mathcal{F}$ is G_L-equivariant, then $\mathcal{F}(n)_{\mathrm{HN}}$ is G_L-equivariant, and the short exact sequences (3.3.1) and (3.3.2) are short exact sequences of G_L-equivariant coherent sheaves.

Proposition 3.3.1 (Fontaine)**.** *Let* $\mathcal{F}$ *be a coherent sheaf over* X^{FF}*. Let* $n \in \mathbf{Z}$*. We have*

$$\mu(\mathcal{F}(n)_{\mathrm{HN}}) = \mu(\mathcal{F}) + n.$$

Moreover, if $\mathcal{F}$ *is semistable, then* $\mathcal{F}(n)_{\mathrm{HN}}$ *is semistable.*

Corollary 3.3.2. *Let* $\mathcal{F}$ *be a coherent sheaf over* X^{FF}*, and let* $(\mathcal{F}_i)_{i\in\{0,\ldots,n\}}$ *be the Harder–Narasimhan filtration of* $\mathcal{F}$*. Let* $n \in \mathbf{Z}$*. Then* $(\mathcal{F}_i(n)_{\mathrm{HN}})_{i\in\{0,\ldots,n\}}$ *is the Harder–Narasimhan filtration of* $\mathcal{F}(n)_{\mathrm{HN}}$*. In particular, the Harder–Narasimhan slopes of* $\mathcal{F}(n)_{\mathrm{HN}}$ *are* $\{\mu_i + n\}_{i\in\{1,\ldots,n\}}$*, where* μ_i *runs over the Harder–Narasimhan slopes of* $\mathcal{F}$*.*

3.4 Almost $\mathbf{C}_p$-representations and coherent sheaves

We recall the relation between almost $\mathbf{C}_p$-representations of G_K and G_K-equivariant coherent sheaves over the Fargues–Fontaine curve established by Fontaine [19].

Let $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)$ be the category of G_K-equivariant coherent sheaves over X^{FF}. We set the following subcategories of $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)$.

- Let $\mathrm{Coh}^{\geq 0}_{X^{\mathrm{FF}}}(G_K)$ be the full subcategory of $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)$ of G_K-equivariant coherent sheaves whose Harder–Narasimhan slopes are all ≥ 0. Note that the category $\mathrm{Coh}^{\geq 0}_{X^{\mathrm{FF}}}(G_K)$ is an exact subcategory of $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)$.
- Let $\mathrm{Coh}^{<0}_{X^{\mathrm{FF}}}(G_K)$ be the full subcategory of $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)$ of G_K-equivariant coherent sheaves whose Harder–Narasimhan slopes are all < 0. Note that the category $\mathrm{Coh}^{<0}_{X^{\mathrm{FF}}}(G_K)$ is an exact subcategory of $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)$.
- Let $\mathrm{Coh}^{0}_{X^{\mathrm{FF}}}(G_K)$ be the full subcategory of $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)$ composed of the G_K-equivariant coherent sheaves semistable of slope 0. Note that the category $\mathrm{Coh}^{0}_{X^{\mathrm{FF}}}(G_K)$ is an abelian subcategory of $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)$.
- Let $\mathrm{Coh}^{+\infty}_{X^{\mathrm{FF}}}(G_K)$ be the full subcategory of $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)$ of G_K-equivariant torsion coherent sheaves. Note that the category $\mathrm{Coh}^{+\infty}_{X^{\mathrm{FF}}}(G_K)$ is an abelian subcategory of $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)$.

The classification of coherent sheaves over X^{FF}, Theorem 3.2.3, implies the following.

Proposition 3.4.1. *The tuple* $(\mathrm{Coh}^{\geq 0}_{X^{\mathrm{FF}}}(G_K), \mathrm{Coh}^{<0}_{X^{\mathrm{FF}}}(G_K))$ *is a torsion pair on* $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)$*.*

Theorem 3.4.2 (Fargues–Fontaine, Fontaine)**.** *There are functors*

$$\begin{aligned} \mathrm{Coh}_{X^{\mathrm{FF}}}(G_K) &\to \mathcal{C}^{\geq 0}(G_K) \\ \mathcal{F} &\mapsto \mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{F}) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Coh}_{X^{\mathrm{FF}}}(G_K) &\to \mathcal{C}^{<0}(G_K) \\ \mathcal{F} &\mapsto \mathrm{H}^1(X^{\mathrm{FF}}, \mathcal{F}) \end{aligned}$$

which induce the following equivalence of categories.

1. *The functor* $\mathcal{F} \mapsto \mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{F})$ *induces an equivalence of exact categories*
$$\mathrm{Coh}^{\geq 0}_{X^{\mathrm{FF}}}(G_K) \xrightarrow{\sim} \mathcal{C}^{\geq 0}(G_K).$$
2. *The functor* $\mathcal{F} \mapsto \mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{F})$ *induces an equivalence of abelian categories*
$$\mathrm{Coh}^{0}_{X^{\mathrm{FF}}}(G_K) \xrightarrow{\sim} \mathcal{C}^{0}(G_K),$$
of which the functor
$$\begin{aligned} \mathcal{C}^0(G_K) &\xrightarrow{\sim} \mathrm{Coh}^0_{X^{\mathrm{FF}}}(G_K) \\ V &\mapsto \mathcal{E}(V) = \mathcal{O}_{X^{\mathrm{FF}}} \otimes_{\mathbf{Q}_p} V = (\mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} V, \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V) \end{aligned}$$
is a quasi-inverse.

3. *The functor* $\mathcal{F} \mapsto \mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{F})$ *induces an equivalence of abelian categories*

$$\mathrm{Coh}^{+\infty}_{X^{\mathrm{FF}}}(G_K) \xrightarrow{\sim} \mathcal{C}^{+\infty}(G_K),$$

of which the functor

$$\mathcal{C}^{+\infty}(G_K) \xrightarrow{\sim} \mathrm{Coh}^{+\infty}_{X^{\mathrm{FF}}}(G_K)$$
$$M^+_{\mathrm{dR}} \mapsto (0, M^+_{\mathrm{dR}})$$

is a quasi-inverse.

4. *The functor* $\mathcal{F} \mapsto \mathrm{H}^1(X^{\mathrm{FF}}, \mathcal{F})$ *induces an equivalence of exact categories*

$$\mathrm{Coh}^{<0}_{X^{\mathrm{FF}}}(G_K) \xrightarrow{\sim} \mathcal{C}^{<0}(G_K).$$

Remark 3.4.3. While the global sections functor does not extend to an equivalence of categories between $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)$ and $\mathcal{C}(G_K)$, Fontaine has proved that it induces an equivalence of triangulated categories between the bounded derived categories

$$\mathrm{D}^b(\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)) \xrightarrow{\sim} \mathrm{D}^b(\mathcal{C}(G_K)). \tag{3.4.1}$$

Fontaine has also proved that the categories $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)$ and $\mathcal{C}(G_K)$ can be recovered from each other as follows.

- The torsion pair $(\mathrm{Coh}^{\geq 0}_{X^{\mathrm{FF}}}(G_K), \mathrm{Coh}^{<0}_{X^{\mathrm{FF}}}(G_K))$ on $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)$ induces a t-structure on $\mathrm{D}^b(\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K))$ whose abelian heart is naturally equivalent to $\mathcal{C}(G_K)$ via the equivalence (3.4.1).
- The torsion pair $(\mathcal{C}^{<0}(G_K), \mathcal{C}^{\geq 0}(G_K))$ on $\mathcal{C}(G_K)$ induces a t-structure on $\mathrm{D}^b(\mathcal{C}(G_K))$ whose abelian heart is naturally equivalent to $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)$ via the equivalence (3.4.1).

We also set the following subcategories of $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)$ and $\mathcal{C}(G_K)$.

- Let $\mathrm{Coh}^{>0}_{X^{\mathrm{FF}}}(G_K)$ be the full subcategory of $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)$ of G_K-equivariant coherent sheaves whose Harder–Narasimhan slopes are all > 0. Note that the category $\mathrm{Coh}^{>0}_{X^{\mathrm{FF}}}(G_K)$ is a full subcategory of $\mathrm{Coh}^{\geq 0}_{X^{\mathrm{FF}}}(G_K)$, and that the category $\mathrm{Coh}^{+\infty}_{X^{\mathrm{FF}}}(G_K)$ is a full subcategory of $\mathrm{Coh}^{>0}_{X^{\mathrm{FF}}}(G_K)$.
- Let $\mathcal{C}^{>0}(G_K)$ be the subcategory of $\mathcal{C}(G_K)$ equivalent to the category $\mathrm{Coh}^{>0}_{X^{\mathrm{FF}}}(G_K)$ via the equivalence of categories $\mathrm{Coh}^{\geq 0}_{X^{\mathrm{FF}}}(G_K) \xrightarrow{\sim} \mathcal{C}^{\geq 0}(G_K)$ from Theorem 3.4.2. Note that the category $\mathcal{C}^{+\infty}(G_K)$ is a full subcategory of $\mathcal{C}^{>0}(G_K)$.

3.5 Almost $\mathbf{C}_p$-representations and p-adic representations

Definition 3.5.1. Let $\mathcal{C}^{\neq 0}(G_K)$ be the full subcategory of $\mathcal{C}(G_K)$ of almost $\mathbf{C}_p$-representation X of G_K such that for all p-adic representation V of G_K, we have

$$\mathrm{Hom}_{\mathcal{C}(G_K)}(X, V) = 0.$$

Proposition 3.5.2. *The tuple $(\mathcal{C}^{\neq 0}(G_K), \mathcal{C}^0(G_K))$ is a torsion pair on $\mathcal{C}(G_K)$. Moreover, if X is an almost $\mathbf{C}_p$-representation of G_K, then there exists a commutative diagram in $\mathcal{C}(G_K)$ whose columns and rows are exact which is unique up to isomorphism*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X^{<0} \arrow[r] \arrow[d, equal] & X^{\neq 0} \arrow[r] \arrow[d] & X^{>0} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X^{<0} \arrow[r] & X \arrow[r] \arrow[d] & X^{\geq 0} \arrow[r] \arrow[d] & 0 \\
 & & X^0 \arrow[r, equal] \arrow[d] & X^0 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd},$$

with $X^{<0}$ an object of $\mathcal{C}^{<0}(G_K)$, $X^{>0}$ an object of $\mathcal{C}^{>0}(G_K)$, $X^{\geq 0}$ an object of $\mathcal{C}^{\geq 0}(G_K)$, X^0 an object of $\mathcal{C}^0(G_K)$, and $X^{\neq 0}$ an object of $\mathcal{C}^{\neq 0}(G_K)$.

Proof. By Definition 3.5.1 of the subcategory $\mathcal{C}^{\neq 0}(G_K)$, the first property in the Definition 2.4.1 of a torsion pair is satisfied. We prove the second one.

Let X be an almost $\mathbf{C}_p$-representation of G_K. Since $(\mathcal{C}^{<0}(G_K), \mathcal{C}^{\geq 0}(G_K))$ is a torsion pair on $\mathcal{C}(G_K)$ by Proposition 2.5.1, there exits a short exact sequence

$$0 \to X^{<0} \to X \to X^{\geq 0} \to 0, \tag{3.5.1}$$

with $X^{<0}$ a coeffective almost $\mathbf{C}_p$-representation of G_K, and $X^{\geq 0}$ an effective almost $\mathbf{C}_p$-representation of G_K. By Theorem 3.4.2, there exists a sheaf $\mathcal{F}^{\geq 0}$ which is an object of $\mathrm{Coh}^{\geq 0}_{X^{\mathrm{FF}}}(G_K)$ and an isomorphism of almost $\mathbf{C}_p$-representations of G_K

$$\mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{F}^{\geq 0}) \xrightarrow{\sim} X^{\geq 0}.$$

Let

$$0 \to \mathcal{F}^{>0} \to \mathcal{F}^{\geq 0} \to \mathcal{F}^0 \to 0, \tag{3.5.2}$$

be the first step of the Harder–Narasimhan filtration of $\mathcal{F}^{\geq 0}$, with $\mathcal{F}^{>0}$ an object of $\mathrm{Coh}^{>0}_{X^{\mathrm{FF}}}(G_K)$, and $\mathcal{F}^0$ an object of $\mathrm{Coh}^{0}_{X^{\mathrm{FF}}}(G_K)$. By Theorem 3.4.2, we have the almost $\mathbf{C}_p$-representations of G_K

$$X^{>0} = \mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{F}^{>0}),$$
$$X^{0} = \mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{F}^{0}),$$

where $X^{>0}$ is an object of $\mathcal{C}^{>0}(G_K)$, and X^0 is a p-adic representation of G_K. By Theorem 3.2.1, the group $\mathrm{H}^1(X^{\mathrm{FF}}, \mathcal{F}^{>0})$ is trivial, and the cohomology of the short exact sequence (3.5.2) gives rise to the short exact sequence

$$0 \to X^{>0} \to X^{\geq 0} \to X^0 \to 0. \tag{3.5.3}$$

Let $X^{\neq 0}$ be the reciprocal image of $X^{>0}$ in X via the short exact sequence (3.5.1). The combination of the short exact sequences (3.5.1) and (3.5.3) yields the commutative diagram in $\mathcal{C}(G_K)$ whose columns and rows are exact

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & X^{<0} \arrow[r] \arrow[d, equal] & X^{\neq 0} \arrow[r] \arrow[d] & X^{>0} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X^{<0} \arrow[r] & X \arrow[r] \arrow[d] & X^{\geq 0} \arrow[r] \arrow[d] & 0 \\ & & X^{0} \arrow[r, equal] \arrow[d] & X^{0} \arrow[d] & \\ & & 0 & 0 & . \end{tikzcd} \tag{3.5.4}$$

We prove that $X^{\neq 0}$ is an object of $\mathcal{C}^{\neq 0}(G_K)$. Let V be a p-adic representation of G_K. On the one hand, since $X^{<0}$ is an object of $\mathcal{C}^{<0}(G_K)$ and the category of p-adic representation $\mathcal{C}^{0}(G_K)$ is a subcategory of $\mathcal{C}^{\geq 0}(G_K)$ by Proposition 2.5.1, and since the tuple $(\mathcal{C}^{<0}(G_K), \mathcal{C}^{\geq 0}(G_K))$ is a torsion pair on $\mathcal{C}(G_K)$ again by Proposition 2.5.1, there is no non-trivial map from $X^{<0}$ to V. On the other hand, the G_K-equivariant vector bundle $\mathcal{E}(V)$ associated with V is semistable of slope 0 by Theorem 3.4.2, therefore, by Proposition 3.4.1 there is no non-trivial map from $\mathcal{F}^{>0}$ to $\mathcal{E}(V)$, and thus, by the equivalence of categories $\mathrm{Coh}^{\geq 0}_{X^{\mathrm{FF}}}(G_K) \xrightarrow{\sim} \mathcal{C}^{\geq 0}(G_K)$ from Theorem 3.4.2, there is no non-trivial map from $X^{>0} = \mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{F}^{>0})$ to $V = \mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{E}(V))$. Therefore, using the short exact sequence

$$0 \to X^{<0} \to X^{\neq 0} \to X^{>0} \to 0$$

extracted from the diagram (3.5.4), we conclude that there is no non-trivial map from $X^{\neq 0}$ to V, and hence, $X^{\neq 0}$ is an object of $\mathcal{C}^{\neq 0}(G_K)$.

Finally, the existence of the short exact sequence

$$0 \to X^{\neq 0} \to X \to X^{0} \to 0$$

extracted from the diagram (3.5.4) with $X^{\neq 0}$ an object of $\mathcal{C}^{\neq 0}(G_K)$ and X^0 an object of $\mathcal{C}^{0}(G_K)$ implies that the second property in the Definition 2.4.1 of a torsion pair is satisfied. ∎

We will also need the following.

Lemma 3.5.3. *If $X^{<0}$ is a coeffective almost $\mathbf{C}_p$-representation of G_K, then there exists a short exact sequence in $\mathcal{C}(G_K)$*

$$0 \to Y^{>0} \to Z^{+\infty} \to X^{<0} \to 0,$$

with $Y^{>0}$ an object of $\mathcal{C}^{>0}(G_K)$, and $Z^{+\infty}$ an object of $\mathcal{C}^{+\infty}(G_K)$.

Proof. By Theorem 3.4.2, there exists an object $\mathcal{F}$ of $\mathrm{Coh}^{<0}_{X^{\mathrm{FF}}}(G_K)$ and an isomorphism of almost $\mathbf{C}_p$-representations

$$\mathrm{H}^1(X^{\mathrm{FF}}, \mathcal{F}) \xrightarrow{\sim} X^{<0}.$$

For $n \in \mathbf{N}$, the Harder–Narasimhan twist $\mathcal{F}(n)_{\mathrm{HN}}$ of $\mathcal{F}$ fits into the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}(n)_{\mathrm{HN}} \to \mathcal{H}(n) \to 0, \tag{3.5.5}$$

with $\mathcal{H}(n)$ the torsion G_K-equivariant coherent sheaf $(0, t^{-n}\mathcal{F}^+_{\mathrm{dR}}/\mathcal{F}^+_{\mathrm{dR}})$. Moreover, for n sufficiently large, the Harder–Narasimhan slopes of $\mathcal{F}(n)_{\mathrm{HN}}$ are all > 0 by Proposition 3.3.1. Hence, by Theorem 3.2.1 and Theorem 3.4.2, the short exact sequence (3.5.5) induces a short exact sequence

$$0 \to \mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{F}(n)_{\mathrm{HN}}) \to \mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{H}(n)) \to \mathrm{H}^1(X^{\mathrm{FF}}, \mathcal{F}) \to 0,$$

and $\mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{F}(n)_{\mathrm{HN}})$ is an object of $\mathcal{C}^{>0}(G_K)$, and $\mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{H}(n))$ is an object of $\mathcal{C}^{+\infty}(G_K)$. ∎

3.6 de Rham vector bundles

We briefly recall the definition of de Rham vector bundles over the Fargues–Fontaine curve [13].

Let Mod_K be the category of finite dimensional K-vector spaces. There is a functor

$$\begin{aligned} \mathrm{Rep}_{\mathbf{B}_{\mathrm{dR}}}(G_K) &\to \mathrm{Mod}_K \\ M_{\mathrm{dR}} &\mapsto M_{\mathrm{dR}}^{G_K}, \end{aligned} \tag{3.6.1}$$

which admits a right adjoint

$$\begin{aligned} \mathrm{Mod}_K &\to \mathrm{Rep}_{\mathbf{B}_{\mathrm{dR}}}(G_K) \\ D &\mapsto \mathbf{B}_{\mathrm{dR}} \otimes_K D. \end{aligned} \tag{3.6.2}$$

A $\mathbf{B}_{\mathrm{dR}}$-representation M_{dR} of G_K is *flat* if $\dim_{\mathbf{B}_{\mathrm{dR}}} M_{\mathrm{dR}} = \dim_K M_{\mathrm{dR}}^{G_K}$. Let $\mathrm{Rep}^{\mathrm{fl.}}_{\mathbf{B}_{\mathrm{dR}}}(G_K)$ be the full subcategory of $\mathrm{Rep}_{\mathbf{B}_{\mathrm{dR}}}(G_K)$ of flat $\mathbf{B}_{\mathrm{dR}}$-representation of G_K. Then the functor (3.6.1) induces an equivalence of categories

$$\mathrm{Rep}^{\mathrm{fl.}}_{\mathbf{B}_{\mathrm{dR}}}(G_K) \xrightarrow{\sim} \mathrm{Mod}_K,$$

of which the functor (3.6.2) is a quasi-inverse.

Let Fil_K be the category of filtered K-vector spaces, that is, the category of finite dimensional K-vector space equipped with a decreasing exhaustive and separated filtration by K-vector subspaces. The weights of a filtered K-vector space $(D, \mathrm{Fil}\, D)$ are the integers $n \in \mathbf{Z}$ such that $\mathrm{Fil}^{-n} D / \mathrm{Fil}^{-n+1} D \neq 0$, and the multiplicity of a weight n is the dimension $\dim_K \mathrm{Fil}^{-n} D / \mathrm{Fil}^{-n+1} D$.

Let $\mathrm{Rep}^{\mathrm{free}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K)$ be the full subcategory of $\mathrm{Rep}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K)$ of $\mathbf{B}^+_{\mathrm{dR}}$-representation of G_K whose underlying $\mathbf{B}^+_{\mathrm{dR}}$-module is free. There is a functor

$$\begin{aligned} \mathrm{Rep}^{\mathrm{free}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K) &\to \mathrm{Fil}_K \\ M^+_{\mathrm{dR}} &\mapsto (M_{\mathrm{dR}}^{G_K}, \{(t^n M^+_{\mathrm{dR}})^{G_K}\}_{n \in \mathbf{Z}}) \end{aligned} \tag{3.6.3}$$

which admits a right adjoint

$$\begin{aligned} \mathrm{Fil}_K &\to \mathrm{Rep}^{\mathrm{free}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K) \\ (D, \mathrm{Fil}\, D) &\mapsto \sum_{n\in\mathbf{Z}} \mathrm{Fil}^n \mathbf{B}_{\mathrm{dR}} \otimes_K \mathrm{Fil}^{-n} D. \end{aligned} \tag{3.6.4}$$

A free $\mathbf{B}^+_{\mathrm{dR}}$-representation M^+_{dR} of G_K is *generically flat* if the $\mathbf{B}_{\mathrm{dR}}$-representation M_{dR} is flat. Let $\mathrm{Rep}^{\mathrm{g.fl.}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K)$ be the full subcategory of $\mathrm{Rep}^{\mathrm{free}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K)$ of generically flat $\mathbf{B}^+_{\mathrm{dR}}$-representation of G_K.

Theorem 3.6.1 (Fargues–Fontaine)**.** *The functor* (3.6.3) *induces an equivalence of categories*

$$\mathrm{Rep}^{\mathrm{g.fl.}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K) \xrightarrow{\sim} \mathrm{Fil}_K,$$

of which the functor (3.6.4) *is a quasi-inverse.*

There is a functor

$$\begin{aligned} \mathbf{D}_{\mathrm{dR}} : \mathrm{Bun}_{X^{\mathrm{FF}}}(G_K) &\to \mathrm{Fil}_K \\ \mathcal{E} &\mapsto (\mathbf{D}_{\mathrm{dR}}(\mathcal{E}), \mathrm{Fil}\, \mathbf{D}_{\mathrm{dR}}(\mathcal{E})), \end{aligned}$$

defined as the composition of the functor (3.6.3) with the functor

$$\begin{aligned} \mathrm{Bun}_{X^{\mathrm{FF}}}(G_K) &\to \mathrm{Rep}^{\mathrm{free}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K) \\ \mathcal{E} &\mapsto \mathcal{E}^+_{\mathrm{dR}}. \end{aligned}$$

A G_K-equivariant vector bundle $\mathcal{E}$ is *de Rham* if $\mathcal{E}^+_{\mathrm{dR}}$ is generically flat, or equivalently, if $\dim_K \mathbf{D}_{\mathrm{dR}}(\mathcal{E}) = \mathrm{rank}\, \mathcal{E}$. The *Hodge–Tate weights* of a de Rham vector bundle $\mathcal{E}$ are the weights of $\mathbf{D}_{\mathrm{dR}}(\mathcal{E})$. We denote by $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}$ the full subcategory of $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)$ of G_K-equivariant de Rham vector bundles over X^{FF}.

Remark 3.6.2. The definition of de Rham vector bundles and Proposition 3.1.2 implies the following characterisation of de Rham vector bundles. A G_K-equivariant coherent sheaf $\mathcal{E}$ is a de Rham vector bundle if and only if $\mathcal{E}^+_{\mathrm{dR}}$ is generically flat.

Proposition 3.6.3. *Let*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

be a short exact sequence of G_K-equivariant vector bundles over X^{FF}. If $\mathcal{E}$ is de Rham, then $\mathcal{E}'$ and $\mathcal{E}''$ are de Rham. Moreover, the set of the Hodge–Tate weights of $\mathcal{E}$ is the union of the sets of the Hodge–Tate weights of $\mathcal{E}'$ and $\mathcal{E}''$.

Remark 3.6.4. The composition of functors

$$\begin{aligned} \mathrm{Rep}_{\mathbf{Q}_p}(G_K) &\to \mathrm{Fil}_K \\ V &\mapsto \mathbf{D}_{\mathrm{dR}}(\mathcal{E}(V)) \end{aligned}$$

is the usual $\mathbf{D}_{\mathrm{dR}}$ functor defined by Fontaine [17]. In particular, a p-adic representation V is de Rham if and only if the vector bundle $\mathcal{E}(V)$ is de Rham. We will write $\mathbf{D}_{\mathrm{dR}}(V)$ instead of $\mathbf{D}_{\mathrm{dR}}(\mathcal{E}(V))$.

4 Cohomology of perfectoid fields

4.1 Cohomology of almost $\mathbf{C}_p$-representations

Let L be an algebraic extension of K. If $\hat{L}$ is a perfectoid field, as a consequence of the classification of G_L-equivariant coherent sheaves over X^{FF} already mentioned in Theorem 3.2.3, Fargues and Fontaine [13, Remarque 9.4.2] have obtained the following.

Theorem 4.1.1 (Fargues–Fontaine)**.** *Let $\mathcal{F}$ be a G_L-equivariant coherent sheaf over X^{FF} whose Harder–Narasimhan slopes are all > 0. If $\hat{L}$ is a perfectoid field, then*

$$\mathrm{Ext}^1_{\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)}(\mathcal{O}_{X^{\mathrm{FF}}},\mathcal{F}) \xrightarrow{\sim} \mathrm{H}^1(L,\mathrm{H}^0(X^{\mathrm{FF}},\mathcal{F})) = 0.$$

The combination of Theorem 3.4.2 and Theorem 4.1.1 immediately implies the following.

Corollary 4.1.2. *Let $X^{>0}$ be an almost $\mathbf{C}_p$-representation which is an object of the subcategory $\mathcal{C}^{>0}(G_K)$. If $\hat{L}$ is a perfectoid field, then the group $\mathrm{H}^1(L,X^{>0})$ is trivial.*

We will use the following repeatedly.

Proposition 4.1.3. *The p-cohomological dimension of a perfectoid field of residue characteristic p is ≤ 1.*

Proof. Let k be a perfectoid field of residue characteristic p. On the one hand, the tilt $k^\flat$ of k is a perfectoid field of characteristic p whose absolute Galois group $G_{k^\flat}$ is canonically isomorphic to G_k ([30, §3]). On the other hand, the p-cohomological dimension of a field characteristic p is ≤ 1 ([32, II §2.2 Proposition 3]). ∎

Lemma 4.1.4. *Let X be a p-adic Banach representation of G_K, and let $\mathcal{X}$ be a G_K-stable lattice in X. If the p-cohomological dimension of L is ≤ 1, then, for each integer $n > 1$, the groups $\mathrm{H}^n(L,\mathcal{X})$ and $\mathrm{H}^n(L,X)$ are trivial.*

Proof. Recall [23, §2] that, since $\mathcal{X}$ is complete and separated for the p-adic topology, for each $n \in \mathbf{N}$, there is a short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{H}^{n-1}(L,\mathcal{X}/p^i\mathcal{X}) \to \mathrm{H}^n(L,\mathcal{X}) \to \varprojlim \mathrm{H}^n(L,\mathcal{X}/p^i\mathcal{X}) \to 0,$$

where we set $\mathrm{H}^{-1}(L,\mathcal{X}/p^i\mathcal{X}) = 0$. Moreover, for each $i \in \mathbf{N}$, the short exact sequence

$$0 \to p^i\mathcal{X}/p^{i+1}\mathcal{X} \to \mathcal{X}/p^{i+1}\mathcal{X} \to \mathcal{X}/p^i\mathcal{X} \to 0$$

induces an exact sequence

$$\mathrm{H}^1(L,\mathcal{X}/p^{i+1}\mathcal{X}) \to \mathrm{H}^1(L,\mathcal{X}/p^i\mathcal{X}) \to \mathrm{H}^2(L,p^i\mathcal{X}/p^{i+1}\mathcal{X}).$$

By hypothesis, for each integers $n > 1$ and $i \in \mathbf{N}$, the groups $\mathrm{H}^n(L,\mathcal{X}/p^i\mathcal{X})$ and $\mathrm{H}^n(L,p^i\mathcal{X}/p^{i+1}\mathcal{X})$ are trivial, which implies that:

- for each integer $n > 1$, the group $\varprojlim \mathrm{H}^n(L,\mathcal{X}/p^i\mathcal{X})$ is trivial,

- for each integer $n > 2$, the group $\varprojlim^1 \mathrm{H}^{n-1}(L, \mathcal{X}/p^i\mathcal{X})$ is trivial,
- for each $i \in \mathbf{N}$, the map $\mathrm{H}^1(L, \mathcal{X}/p^{i+1}\mathcal{X}) \to \mathrm{H}^1(L, \mathcal{X}/p^i\mathcal{X})$ is surjective, and thus, the group $\varprojlim^1 \mathrm{H}^1(L, \mathcal{X}/p^i\mathcal{X})$ is also trivial.

Therefore, for each integer $n > 1$, the group $\mathrm{H}^n(L, \mathcal{X})$ is trivial.

Since $X/\mathcal{X}$ is discrete, the short exact sequence

$$0 \to \mathcal{X} \to X \to X/\mathcal{X} \to 0$$

induces a long exact sequence

$$\cdots \to \mathrm{H}^n(L, \mathcal{X}) \to \mathrm{H}^n(L, X) \to \mathrm{H}^n(L, X/\mathcal{X}) \to \mathrm{H}^{n+1}(L, \mathcal{X}) \to \cdots.$$

For each integer $n > 1$, we have proved that the group $\mathrm{H}^n(L, \mathcal{X})$ is trivial, and by hypothesis, the group $\mathrm{H}^n(L, X/\mathcal{X})$ is trivial. Therefore, for each integer $n > 1$, the group $\mathrm{H}^n(L, X)$ is trivial. ∎

Remark 4.1.5. A short exact sequence in $\mathcal{C}(G_K)$

$$0 \to Z \to X \to Y \to 0 \tag{4.1.1}$$

induces a long exact sequence

$$\cdots \to \mathrm{H}^n(L, Z) \to \mathrm{H}^n(L, X) \to \mathrm{H}^n(L, Y) \to \mathrm{H}^{n+1}(L, Z) \to \cdots.$$

Indeed, by Theorem 2.1.1, the category $\mathcal{C}(G_K)$ is an abelian subcategory of $\mathcal{B}(G_K)$. In particular, each morphism of Banach spaces in the sequence (4.1.1) is strict. Moreover, there exists a section, as topological spaces, of the surjective morphism $X \to Y$ (see for instance [10, Proposition I.1.5 (iii)]). Therefore, the short exact sequence (4.1.1) induces long exact sequences in Galois cohomology [34, §2].

Proposition 4.1.6. *Let X be an almost $\mathbf{C}_p$-representation of G_K. If $\hat{L}$ is a perfectoid field, then the short exact sequence*

$$0 \to X^{\neq 0} \to X \to X^0 \to 0$$

induces an isomorphism

$$\mathrm{H}^1(L, X) \xrightarrow{\sim} \mathrm{H}^1(L, X^0).$$

Proof. The combination of Proposition 4.1.3, Lemma 4.1.4 and Remark 4.1.5 yields an exact sequence

$$\mathrm{H}^1(L, X^{\neq 0}) \to \mathrm{H}^1(L, X) \to \mathrm{H}^1(L, X^0) \to 0.$$

We prove that the group $\mathrm{H}^1(L, X^{\neq 0})$ is trivial. By Proposition 3.5.2, there exists a short exact sequence

$$0 \to X^{<0} \to X^{\neq 0} \to X^{>0} \to 0, \tag{4.1.2}$$

with $X^{<0}$ an object of $\mathcal{C}^{<0}(G_K)$, and $X^{>0}$ an object of $\mathcal{C}^{>0}(G_K)$. By Proposition 4.1.3, Lemma 4.1.4 and Remark 4.1.5, the short exact sequence (4.1.2) induces an exact sequence

$$\mathrm{H}^1(L, X^{<0}) \to \mathrm{H}^1(L, X^{\neq 0}) \to \mathrm{H}^1(L, X^{>0}) \to 0. \tag{4.1.3}$$

Moreover, by Lemma 3.5.3, there exists a short exact sequence

$$0 \to Y^{>0} \to Z^{+\infty} \to X^{<0} \to 0, \tag{4.1.4}$$

with $Y^{>0}$ an object of $\mathcal{C}^{>0}(G_K)$, and $Z^{+\infty}$ an object of $\mathcal{C}^{+\infty}(G_K)$. Again by Proposition 4.1.3, Lemma 4.1.4 and Remark 4.1.5, the short exact sequence (4.1.4) induces an exact sequence

$$\mathrm{H}^1(L, Y^{>0}) \to \mathrm{H}^1(L, Z^{+\infty}) \to \mathrm{H}^1(L, X^{<0}) \to 0. \tag{4.1.5}$$

Finally, by Corollary 4.1.2, the groups $\mathrm{H}^1(L, Z^{+\infty})$ and $\mathrm{H}^1(L, X^{>0})$ are trivial. Thus, using the exact sequences (4.1.3) and (4.1.5), we conclude that the groups $\mathrm{H}^1(L, X^{<0})$ and $\mathrm{H}^1(L, X^{\neq 0})$ are trivial. ∎

Lemma 4.1.7. *Let*

$$0 \to Z \to X \to Y \to 0$$

be a short exact sequence in $\mathcal{C}(G_K)$, and let $\mathcal{Z}$ be a G_K-stable lattice in Z. If the p-cohomological dimension of L is ≤ 1, then the short exact sequence of topological G_K-modules

$$0 \to \mathcal{Z} \to X \to X/\mathcal{Z} \to 0$$

induces a surjective map

$$\mathrm{H}^1(L, X) \to \mathrm{H}^1(L, X/\mathcal{Z}) \to 0$$

Proof. We consider the diagram of topological G_K-modules whose rows and columns are exact

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{Z} \arrow[r, equal] \arrow[d] & \mathcal{Z} \arrow[d] & & \\
0 \arrow[r] & Z \arrow[r] \arrow[d, "g"] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z/\mathcal{Z} \arrow[r] \arrow[d] & X/\mathcal{Z} \arrow[r, "f'"] \arrow[d] & Y \arrow[r] & 0 \\
 & 0 & 0 & & .
\end{tikzcd} \tag{4.1.6}$$

The maps f, f' and g each admits a continuous section as morphisms of topological spaces. Hence, by Lemma 4.1.4, the diagram (4.1.6) induces the diagram whose rows and columns are exact

$$\begin{array}{ccccccccc}
Y^{G_L} & \longrightarrow & \mathrm{H}^1(L,Z) & \longrightarrow & \mathrm{H}^1(L,X) & \longrightarrow & \mathrm{H}^1(L,Y) & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \| & & \\
Y^{G_L} & \longrightarrow & \mathrm{H}^1(L,Z/\mathcal{Z}) & \longrightarrow & \mathrm{H}^1(L,X/\mathcal{Z}) & \longrightarrow & \mathrm{H}^1(L,Y) & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & & ,
\end{array}$$

from which the statement follows. ∎

The combination of Proposition 4.1.3, Proposition 4.1.6 and Lemma 4.1.7 implies the following.

Corollary 4.1.8. *Let*

$$0 \to Z \to X \to Y \to 0$$

be a short exact sequence in $\mathcal{C}(G_K)$, and let $\mathcal{Z}$ be a G_K-stable lattice in Z. Let $\mathcal{Z}^{(0)}$ be the image of $\mathcal{Z}$ in X^0, and let $\mathcal{Z}^{(\neq 0)} = \mathcal{Z} \cap X^{\neq 0}$. If $\hat{L}$ is a perfectoid field, then the short exact sequence of topological G_K-modules

$$0 \to X^{\neq 0}/\mathcal{Z}^{(\neq 0)} \to X/\mathcal{Z} \to X^0/\mathcal{Z}^{(0)} \to 0$$

induces an isomorphism

$$\mathrm{H}^1(L, X/\mathcal{Z}) \xrightarrow{\sim} \mathrm{H}^1(L, X^0/\mathcal{Z}^{(0)}).$$

Moreover, for each integer $n > 1$, the group $\mathrm{H}^n(L, X/\mathcal{Z})$ is trivial.

4.2 Cohomology of maximal discrete Galois submodules

Notation 4.2.1. If M is a topological G_K-module, then we denote by M_δ the discrete G_K-module

$$M_\delta = \varinjlim_{\mathrm{res}, K'} \mathrm{H}^0(K', M),$$

where K' runs over all the finite extensions of K, and the transition morphisms are the restriction maps.

Let

$$0 \to Z \to X \to Y \to 0$$

be a short exact sequence in $\mathcal{C}(G_K)$, and let $\mathcal{Z}$ be a G_K-stable lattice in Z.

Lemma 4.2.2. *The short exact sequence of topological G_K-modules*

$$0 \to Z/\mathcal{Z} \to X/\mathcal{Z} \to Y \to 0$$

induces a short exact sequence of discrete G_K-modules

$$0 \to Z/\mathcal{Z} \to (X/\mathcal{Z})_\delta \to Y_\delta \to 0.$$

In particular, there is a commutative diagram of topological G_K-modules with exact rows

$$\begin{tikzcd} 0 \arrow[r] & Z/\mathcal{Z} \arrow[r] & X/\mathcal{Z} \arrow[r] & Y \arrow[r] & 0 \\ 0 \arrow[r] & Z/\mathcal{Z} \arrow[r] \arrow[u, equal] & (X/\mathcal{Z})_\delta \arrow[r] \arrow[u] & Y_\delta \arrow[r] \arrow[u] & 0. \end{tikzcd}$$

Proof. The short exact sequence of topological G_K-modules

$$0 \to Z/\mathcal{Z} \to X/\mathcal{Z} \to Y \to 0$$

induces a long exact sequence for each finite extension K' of K

$$0 \to \mathrm{H}^0(K', Z/\mathcal{Z}) \to \mathrm{H}^0(K', X/\mathcal{Z}) \to \mathrm{H}^0(K', Y) \to \mathrm{H}^1(K', Z/\mathcal{Z}) \to \cdots. \tag{4.2.1}$$

The long exact sequences (4.2.1) yields the long exact sequence

$$0 \to (Z/\mathcal{Z})_\delta \to (X/\mathcal{Z})_\delta \to Y_\delta \to \varinjlim_{\mathrm{res}, K'} \mathrm{H}^1(K', Z/\mathcal{Z}) \to \cdots,$$

where K' runs over all the finite extensions of K, and the transition morphisms are the restriction maps. Since $Z/\mathcal{Z}$ is a discrete G_K-module, we have [32, I §2.2 Proposition 8]

$$\varinjlim_{\mathrm{res}, K'} \mathrm{H}^n(K', Z/\mathcal{Z}) = \begin{cases} (Z/\mathcal{Z})_\delta = Z/\mathcal{Z} & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

∎

Let L be an algebraic extension of K.

Lemma 4.2.3. *The short exact sequence of discrete G_K-modules*

$$0 \to Z/\mathcal{Z} \to (X/\mathcal{Z})_\delta \to Y_\delta \to 0$$

induces an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^0(L, Z/\mathcal{Z}) \arrow[r] & \mathrm{H}^0(L, (X/\mathcal{Z})_\delta) \arrow[r] & \mathrm{H}^0(L, Y_\delta) \arrow[dll, rounded corners] \\ & \mathrm{H}^1(L, Z/\mathcal{Z}) \arrow[r] & \mathrm{H}^1(L, (X/\mathcal{Z})_\delta) \arrow[r] & 0 \end{tikzcd},$$

and, for each integer $n > 1$, an isomorphism

$$\mathrm{H}^n(L, Z/\mathcal{Z}) \xrightarrow{\sim} \mathrm{H}^n(L, (X/\mathcal{Z})_\delta).$$

Proof. The short exact sequence of discrete G_K-modules

$$0 \to Z/\mathcal{Z} \to (X/\mathcal{Z})_\delta \to Y_\delta \to 0$$

induces a long exact sequence

$$\cdots \to \mathrm{H}^n(L, Z/\mathcal{Z}) \to \mathrm{H}^n(L, (X/\mathcal{Z})_\delta) \to \mathrm{H}^n(L, Y_\delta) \to \mathrm{H}^{n+1}(L, Z/\mathcal{Z}) \to \cdots.$$

Furthermore, since Y_δ is a uniquely divisible discrete G_K-module, for each integer $n \geq 1$, the group $\mathrm{H}^n(L, Y_\delta)$ is trivial since it is torsion [32, I §2.2 Corollaire 3] and uniquely divisible. ∎

By Lemma 4.2.2 and Lemma 4.2.3, there exists a morphism

$$\xi \,:\, \mathrm{H}^1(L,(X/\mathcal{Z})_\delta) \to \mathrm{H}^1(L,X/\mathcal{Z}),$$

and a commutative diagram with exact rows

$$\begin{tikzcd}
Y^{G_L} \arrow[r] & \mathrm{H}^1(L,Z/\mathcal{Z}) \arrow[r] & \mathrm{H}^1(L,X/\mathcal{Z}) \arrow[r] & \mathrm{H}^1(L,Y) \\
Y_\delta^{G_L} \arrow[r] \arrow[u] & \mathrm{H}^1(L,Z/\mathcal{Z}) \arrow[r] \arrow[u, equal] & \mathrm{H}^1(L,(X/\mathcal{Z})_\delta) \arrow[r] \arrow[u, "\xi"] & 0. \arrow[u]
\end{tikzcd} \tag{4.2.2}$$

In particular, the map ξ induces a morphism

$$\mathrm{H}^1(L,(X/\mathcal{Z})_\delta) \to \mathrm{Ker}\left(\mathrm{H}^1(L,X/\mathcal{Z}) \to \mathrm{H}^1(L,Y)\right).$$

Note that $\mathrm{Ker}\left(\mathrm{H}^1(L,X/\mathcal{Z}) \to \mathrm{H}^1(L,Y)\right)$ is the torsion subgroup of $\mathrm{H}^1(L,X/\mathcal{Z})$.

We consider Y^{G_L} endowed with the subspace topology from Y.

Proposition 4.2.4. *If $(Y_\delta)^{G_L}$ is dense in Y^{G_L}, then the map ξ induces an isomorphism*

$$\mathrm{H}^1(L,(X/\mathcal{Z})_\delta) \xrightarrow{\sim} \mathrm{Ker}\left(\mathrm{H}^1(L,X/\mathcal{Z}) \to \mathrm{H}^1(L,Y)\right).$$

Proof. Recall [29, Appendix A] that if G is a locally compact and separated topological group, and if M is a topological G-module, then the compact-open topology on the continuous cochains induces a structure of topological groups on each abelian group $\mathrm{H}^n(G,M)$, $n \in \mathbf{N}$, which satisfy the following properties.

1. If
$$0 \to M' \to M \to M'' \to 0$$
is a short exact sequence of topological G-modules such that the topology of M' is the subspace topology from M, the topology of M'' is the quotient topology from M, and there exists a continuous section of the projection of M on M'' as topological space, then it induces a sequence of topological groups
$$\cdots \to \mathrm{H}^n(G,M') \to \mathrm{H}^n(G,M) \to \mathrm{H}^n(G,M'') \to \mathrm{H}^{n+1}(G,M') \to \cdots$$
whose underlying sequence of abelian groups is exact.

2. If G is compact and M is a discrete G-module, then, for each $n \in \mathbf{N}$, the topological group $\mathrm{H}^n(L,M)$ is discrete.

Therefore, we can consider the top exact sequence

$$Y^{G_L} \to \mathrm{H}^1(L,Z/\mathcal{Z}) \to \mathrm{H}^1(L,X/\mathcal{Z}) \to \mathrm{H}^1(L,Y)$$

of the diagram (4.2.2) as a sequence of topological abelian groups in which the group $\mathrm{H}^1(L,Z/\mathcal{Z})$ is discrete. If $Y_\delta^{G_L}$ is dense in Y^{G_L}, then, by continuity, the images of $Y_\delta^{G_L}$ and Y^{G_L} in the discrete group $\mathrm{H}^1(L,Z/\mathcal{Z})$ coincides. We conclude using the commutativity of the diagram (4.2.2). ∎

The combination of Corollary 4.1.8 and Proposition 4.2.4 yields the following.

Corollary 4.2.5. *If $\hat{L}$ is a perfectoid field and if $(Y_\delta)^{G_L}$ is dense in Y^{G_L}, then there exists a short exact sequence*

$$0 \to \mathrm{H}^1(L, (X/\mathcal{Z})_\delta) \to \mathrm{H}^1(L, X^0/\mathcal{Z}^{(0)}) \to \mathrm{H}^1(L, Y^0) \to 0.$$

5 Truncation of the Hodge–Tate filtration

Let $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}^{\leq 0}$ be the full subcategory of $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}$ of de Rham G_K-equivariant vector bundles over X^{FF} whose Hodge–Tate weights are all ≤ 0. In [29, §2], we have defined and studied the functor

$$\begin{aligned} \tau_{\mathrm{dR}}^{\leq 0} \,:\, \mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}} &\to \mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}^{\leq 0} \\ \mathcal{E} &\mapsto \mathcal{E}_+ \end{aligned}$$

which is a left adjoint to the forgetful functor from $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}^{\leq 0}$ to $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}$, and which associates with a vector bundle $\mathcal{E} = (\mathcal{E}_{\mathrm{e}}, \mathcal{E}_{\mathrm{dR}}^+, \iota_{\mathcal{E}})$ the vector bundle

$$\mathcal{E}_+ = (\mathcal{E}_{\mathrm{e}}, \mathcal{E}_{\mathrm{dR}}^+ + \mathbf{B}_{\mathrm{dR}}^+ \otimes_K \mathbf{D}_{\mathrm{dR}}(\mathcal{E}), \iota_{\mathcal{E}}).$$

Let $n \geq 1$ be an integer. Let $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}^{\leq 0, > n}$ be the full subcategory of $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}$ of de Rham G_K-equivariant vector bundles over X^{FF} whose Hodge–Tate weights are all in the set $\mathbf{Z} \setminus [1, n]$. In this section 5, we will define and study the functor

$$\begin{aligned} \tau_{\mathrm{dR}}^{\leq 0, > n} \,:\, \mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}} &\to \mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}^{\leq 0, > n} \\ \mathcal{E} &\mapsto \mathcal{E}_+^n \end{aligned}$$

which is a left adjoint to the forgetful functor from $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}^{\leq 0, > n}$ to $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}$, and which associates with a vector bundle $\mathcal{E} = (\mathcal{E}_{\mathrm{e}}, \mathcal{E}_{\mathrm{dR}}^+, \iota_{\mathcal{E}})$ the vector bundle

$$\mathcal{E}_+^n = (\mathcal{E}_{\mathrm{e}}, \mathcal{E}_{\mathrm{dR}}^+ + \mathbf{B}_{\mathrm{dR}}^+ \otimes_K \mathrm{Fil}^{-n} \mathbf{D}_{\mathrm{dR}}(\mathcal{E}), \iota_{\mathcal{E}}).$$

The construction of the functor $\tau_{\mathrm{dR}}^{\leq 0, > n}$ is similar to the one of $\tau_{\mathrm{dR}}^{\leq 0}$. Therefore, we will omit some details and refer the reader to [29]. In particular, this section 5 purposefully follows the same structure as [29, §2] for the reader's convenience.

5.1 Modification of filtered vector spaces

Let $n \geq 1$ be an integer. Let $\mathrm{Fil}_K^{\leq 0, > n}$ be the full subcategory of Fil_K of filtered K-vector spaces whose weights are all in the set $\mathbf{Z} \setminus [1, n]$.

We define the functor

$$\begin{aligned} \tau_{\mathrm{Fil}}^{\leq 0, > n} \,:\, \mathrm{Fil}_K &\to \mathrm{Fil}_K^{\leq 0, > n} \\ (D, \mathrm{Fil}\, D) &\mapsto (D, \mathrm{Fil}_{+,n} D) \end{aligned}$$

which associates with a filtered K-vector space $(D, \mathrm{Fil}\, D)$ the filtered K-vector space $(D, \mathrm{Fil}_{+,n} D)$ where

$$\mathrm{Fil}^i_{+,n} D = \begin{cases} \mathrm{Fil}^i D & \text{if } i \leq -n \\ \mathrm{Fil}^{-n} D & \text{if } i \in [-n, 0] \\ \mathrm{Fil}^i D & \text{if } i > 0. \end{cases}$$

Note that the identity map on D induces a morphism of filtered K-vector spaces $\eta_D : (D, \mathrm{Fil}\, D) \to (D, \mathrm{Fil}_{+,n} D)$.

Proposition 5.1.1. *The functor $\tau^{\leq 0, >n}_{\mathrm{Fil}}$ is exact and left adjoint to the forgetful functor from $\mathrm{Fil}^{\leq 0, >n}_K$ to Fil_K. Moreover, we have the following properties.*

1. *Let $(D, \mathrm{Fil}\, D)$ be a filtered K-vector space. The morphism η_D is the universal morphism from $(D, \mathrm{Fil}\, D)$ to the forgetful functor from $\mathrm{Fil}^{\leq 0, >n}_K$ to Fil_K.*

2. *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Fil}_K & \xrightarrow{\tau^{\leq 0, >n}_{\mathrm{Fil}}} & \mathrm{Fil}^{\leq 0, >n}_K \\ \downarrow & & \downarrow \\ \mathrm{Mod}_K & \xrightarrow{\mathrm{id}} & \mathrm{Mod}_K, \end{array}$$

where the vertical arrows are the forgetful functor $(D, \mathrm{Fil}\, D) \mapsto D$, and the bottom arrow is the identity functor.

Proof. The statement is proved similarly to [29, Proposition 2.1.4, Remark 2.1.3, and Corollary 2.1.5]. ∎

Lemma 5.1.2. *Let $(D, \mathrm{Fil}\, D)$ be a filtered K-vector space. Then*

$$\sum_{i \in \mathbf{Z}} \mathrm{Fil}^i \mathbf{B}^+_{\mathrm{dR}} \otimes_K \mathrm{Fil}^{-i}_{+,n} D = \left(\sum_{i \in \mathbf{Z}} \mathrm{Fil}^i \mathbf{B}^+_{\mathrm{dR}} \otimes_K \mathrm{Fil}^{-i} D \right) + \mathbf{B}^+_{\mathrm{dR}} \otimes_K \mathrm{Fil}^{-n} D.$$

Proof. The statement follows from computations similar to [29, Lemma 2.1.6]. By definition, we have

$$\begin{aligned} \sum_{i \in \mathbf{Z}} \mathrm{Fil}^i \mathbf{B}^+_{\mathrm{dR}} \otimes_K \mathrm{Fil}^{-i}_{+,n} D = & \sum_{i < 0} \mathrm{Fil}^i \mathbf{B}^+_{\mathrm{dR}} \otimes_K \mathrm{Fil}^{-i} D \\ & + \sum_{i \in [0,n]} \mathrm{Fil}^i \mathbf{B}^+_{\mathrm{dR}} \otimes_K \mathrm{Fil}^{-n} D \\ & + \sum_{i \geq n} \mathrm{Fil}^i \mathbf{B}^+_{\mathrm{dR}} \otimes_K \mathrm{Fil}^{-i} D \end{aligned}$$

On the one hand, since $\mathrm{Fil}^i \mathbf{B}^+_{\mathrm{dR}} \otimes_K \mathrm{Fil}^{-i} D \subseteq \mathrm{Fil}^i \mathbf{B}^+_{\mathrm{dR}} \otimes_K \mathrm{Fil}^{-n} D$ for each $i \in [0, n]$, we have

$$\begin{aligned} \sum_{i \in \mathbf{Z}} \mathrm{Fil}^i \mathbf{B}^+_{\mathrm{dR}} \otimes_K \mathrm{Fil}^{-i}_{+,n} D = & \sum_{i \in \mathbf{Z}} \mathrm{Fil}^i \mathbf{B}^+_{\mathrm{dR}} \otimes_K \mathrm{Fil}^{-i} D \\ & + \sum_{i \in [0,n]} \mathrm{Fil}^i \mathbf{B}^+_{\mathrm{dR}} \otimes_K \mathrm{Fil}^{-n} D. \end{aligned}$$

On the other hand, we have

$$\sum_{i\in[0,n]} \mathrm{Fil}^i \mathbf{B}^+_{\mathrm{dR}} \otimes_K \mathrm{Fil}^{-n} D = \mathbf{B}^+_{\mathrm{dR}} \otimes_K \mathrm{Fil}^{-n} D.$$

∎

Let $\mathrm{Rep}^{\text{g.fl.}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K)^{\leq 0,>n}$ be the subcategory of $\mathrm{Rep}^{\text{g.fl.}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K)$ equivalent to the subcategory $\mathrm{Fil}_K^{\leq 0,>n}$ of Fil_K via the equivalence $\mathrm{Rep}^{\text{g.fl.}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K) \xrightarrow{\sim} \mathrm{Fil}_K$ from Theorem 3.6.1. By Lemma 5.1.2, the composition of the functor $\tau^{\leq 0,>n}_{\mathrm{Fil}}$ with the equivalences of categories $\mathrm{Rep}^{\text{g.fl.}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K) \xrightarrow{\sim} \mathrm{Fil}_K$ and $\mathrm{Rep}^{\text{g.fl.}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K)^{\leq 0,>n} \xrightarrow{\sim} \mathrm{Fil}_K^{\leq 0,>n}$ then yields a functor

$$\begin{aligned} \tau^{\leq 0,>n}_{\mathrm{dR}} : \mathrm{Rep}^{\text{g.fl.}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K) &\to \mathrm{Rep}^{\text{g.fl.}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K)^{\leq 0,>n} \\ M^+_{\mathrm{dR}} &\mapsto M^+_{\mathrm{dR}} + \mathbf{B}^+_{\mathrm{dR}} \otimes_K (t^{-n} M^+_{\mathrm{dR}})^{G_K}. \end{aligned}$$

Proposition 5.1.1 implies the following.

Proposition 5.1.3. *The functor $\tau^{\leq 0,>n}_{\mathrm{dR}}$ is exact and left adjoint to the forgetful functor from $\mathrm{Rep}^{\text{g.fl.}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K)^{\leq 0,>n}$ to $\mathrm{Rep}^{\text{g.fl.}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K)$. Moreover, we have the following properties.*

1. *Let M^+_{dR} be a generically flat $\mathbf{B}^+_{\mathrm{dR}}$-representation of G_K. The inclusion morphism $M^+_{\mathrm{dR}} \subset M^+_{\mathrm{dR}} + \mathbf{B}^+_{\mathrm{dR}} \otimes_K (t^{-n} M^+_{\mathrm{dR}})^{G_K}$ is the universal morphism from M^+_{dR} to the forgetful functor from $\mathrm{Rep}^{\text{g.fl.}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K)^{\leq 0,>n}$ to $\mathrm{Rep}^{\text{g.fl.}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K)$.*

2. *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Rep}^{\text{g.fl.}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K) & \xrightarrow{\tau^{\leq 0,>n}_{\mathrm{dR}}} & \mathrm{Rep}^{\text{g.fl.}}_{\mathbf{B}^+_{\mathrm{dR}}}(G_K)^{\leq 0,>n} \\ \downarrow & & \downarrow \\ \mathrm{Rep}^{\text{fl.}}_{\mathbf{B}_{\mathrm{dR}}}(G_K) & \xrightarrow{\mathrm{id}} & \mathrm{Rep}^{\text{fl.}}_{\mathbf{B}_{\mathrm{dR}}}(G_K), \end{array}$$

   where the vertical arrows are the functor of extension of scalars, and the bottom arrow is the identity functor.

5.2 Modification of de Rham vector bundles

Let $n \geq 1$ be an integer. Let $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)^{\leq 0,>n}_{\mathrm{dR}}$ be the full subcategory of $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}$ of G_K-equivariant de Rham vector bundles over X^{FF} whose Hodge–Tate weights are all in the set $\mathbf{Z} \setminus [1,n]$.

By Remark 3.6.2 and Proposition 5.1.3, the functor $\tau^{\leq 0,>n}_{\mathrm{dR}}$ together with the identity functor on $\mathrm{Rep}_{\mathbf{B}_{\mathrm{e}}}(G_K)$ induces a functor

$$\begin{aligned} \tau^{\leq 0,>n}_{\mathrm{HT}} : \mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}} &\to \mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)^{\leq 0,>n}_{\mathrm{dR}} \\ \mathcal{E} &\mapsto \mathcal{E}^n_+ \end{aligned}$$

which associates with a de Rham vector bundle $\mathcal{E} = (\mathcal{E}_{\mathrm{e}}, \mathcal{E}^+_{\mathrm{dR}}, \iota_{\mathcal{E}})$ the de Rham vector bundle

$$\mathcal{E}^n_+ = (\mathcal{E}_{\mathrm{e}}, \mathcal{E}^+_{\mathrm{dR}} + \mathbf{B}^+_{\mathrm{dR}} \otimes_K \mathrm{Fil}^{-n} \mathbf{D}_{\mathrm{dR}}(\mathcal{E}), \iota_{\mathcal{E}}).$$

Proposition 5.2.1. *The functor $\tau_{\mathrm{HT}}^{\leq 0,>n}$ is exact and left adjoint to the forgetful functor from $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}^{\leq 0,>n}$ to $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}$. Moreover, we have the following properties.*

1. *Let $\mathcal{E}$ be a G_K-equivariant de Rham vector bundle. The inclusion map $\mathcal{E} \to \mathcal{E}_+^n$ is the universal morphism from $\mathcal{E}$ to the forgetful functor from $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}^{\leq 0,>n}$ to $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}$.*
2. *The composition of the functor $\tau_{\mathrm{HT}}^{\leq 0,>n}$ with the forgetful functor*
$$\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}^{\leq 0,>n} \to \mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}^{\leq 0,>n} \xrightarrow{\tau_{\mathrm{HT}}^{\leq 0,>n}} \mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}^{\leq 0,>n}$$
is isomorphic to the identity functor on $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}^{\leq 0,>n}$.

Proof. All statements except the last follow from Proposition 5.1.3. The last statement is a formal consequence of the adjunction property of $\tau_{\mathrm{HT}}^{\leq 0,>n}$ and the fully faithfulness of the forgetful functor (see [25, IV §3 Theorem 1]). ∎

5.3 Hodge–Tate and Harder–Narasimhan filtrations

Let $n \geq 1$ be an integer. Let V be a de Rham p-adic representation of G_K. The vector bundle $\mathcal{E}(V)$ associated with V is G_K-equivariant and de Rham, and we denote its modification by $\tau_{\mathrm{dR}}^{\leq 0,>n}$ by

$$\mathcal{E}_+^n(V) = (\mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} V, \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V + \mathbf{B}_{\mathrm{dR}}^+ \otimes_K \mathrm{Fil}^{-n} \mathbf{D}_{\mathrm{dR}}(V), \iota_{\mathcal{E}(V)}).$$

The inclusion map of $\mathcal{E}(V)$ in $\mathcal{E}_+^n(V)$ induces a short exact sequence in $\mathrm{Coh}_{X^{\mathrm{FF}}}(G_K)$

$$0 \to \mathcal{E}(V) \to \mathcal{E}_+^n(V) \to \mathcal{F}_+^n(V) \to 0, \tag{5.3.1}$$

with

$$\mathcal{F}_+^n(V) = \left(0, \frac{\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V + \mathbf{B}_{\mathrm{dR}}^+ \otimes_K \mathrm{Fil}^{-n} \mathbf{D}_{\mathrm{dR}}(V)}{\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V}\right).$$

Lemma 5.3.1. *The Harder–Narasimhan slopes of $\mathcal{E}_+^n(V)$ are all ≥ 0.*

Proof. The short exact sequence (5.3.1) induces the cohomological exact sequence

$$\mathrm{H}^1(X^{\mathrm{FF}}, \mathcal{E}(V)) \to \mathrm{H}^1(X^{\mathrm{FF}}, \mathcal{E}_+(V)) \to \mathrm{H}^1(X^{\mathrm{FF}}, \mathcal{F}_+(V)) \to 0. \tag{5.3.2}$$

Since $\mathcal{E}(V)$ and $\mathcal{F}_+(V)$ are semi-stable of slopes 0 and $+\infty$ respectively, the groups $\mathrm{H}^1(X^{\mathrm{FF}}, \mathcal{E}(V))$ and $\mathrm{H}^1(X^{\mathrm{FF}}, \mathcal{F}_+(V))$ are trivial by Proposition 3.2.1. Therefore, the exact sequence (5.3.2) forces the group $\mathrm{H}^1(X^{\mathrm{FF}}, \mathcal{E}_+(V))$ to vanish, and the lemma follows from another application of Proposition 3.2.1. ∎

Notation 5.3.2. We denote by

$$0 \to V_{0,n} \to V \to V^{\leq 0,>n} \to 0$$

the short exact sequence of p-adic representations of G_K where $V^{\leq 0,>n}$ is the maximal quotient representation of V whose Hodge–Tate weights are all in the set $\mathbf{Z} \setminus [1, n]$.

Lemma 5.3.3. *There exists no non-trivial quotient of $V_{0,n}$ whose Hodge–Tate weights all in $\mathbf{Z} \setminus [1, n]$.*

Proof. Let U be a subrepresentation of $V_{0,n}$ such that the Hodge–Tate weights of $V_{0,n}/U$ are all in $\mathbf{Z} \setminus [1, n]$. Then there is a short exact sequence of p-adic representation of G_K

$$0 \to V_{0,n}/U \to V/U \to V^{\leq 0,>n} \to 0.$$

By Proposition 3.6.3, the representation V/U and its Hodge–Tate weights are all in $\mathbf{Z} \setminus [1, n]$. By maximality of $V^{\leq 0,>n}$, we then have $U = V_{0,n}$. ∎

By Theorem 3.4.2, the short exact sequence of de Rham representations

$$0 \to V_{0,n} \to V \to V^{\leq 0,>n} \to 0$$

induces a short exact sequence in $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}$

$$0 \to \mathcal{E}(V_{0,n}) \to \mathcal{E}(V) \to \mathcal{E}(V^{\leq 0,>n}) \to 0,$$

which in turns, by exactness of $\tau_{\mathrm{HT}}^{\leq 0,>n}$ from Proposition 5.2.1, induces a short exact sequence in $\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}^{\leq 0,>n}$

$$0 \to \mathcal{E}_+^n(V_{0,n}) \to \mathcal{E}_+^n(V) \to \mathcal{E}_+^n(V^{\leq 0,>n}) \to 0.$$

Proposition 5.3.4. *The short exact sequence*

$$0 \to \mathcal{E}_+^n(V_{0,n}) \to \mathcal{E}_+^n(V) \to \mathcal{E}_+^n(V^{\leq 0,>n}) \to 0$$

is the first step of the Harder–Narasimhan filtration of $\mathcal{E}_+^n(V)$ with

$$\mathcal{E}_+^n(V^{\leq 0,>n}) = \mathcal{E}(V^{\leq 0,>n})$$

semistable of slope 0*.*

Proof. The statement is proved similarly to [29, Proposition 2.3.5 and Corollary 2.3.6]. By Lemma 5.3.1, the Harder–Narasimhan slopes of each vector bundle in the short exact sequences are all ≥ 0. Moreover, since the Hodge–Tate weights of $V^{\leq 0,>n}$ are all in $\mathbf{Z} \setminus [1, n]$, by Proposition 5.2.1, we have $\mathcal{E}_+^n(V^{\leq 0,>n}) = \mathcal{E}(V^{\leq 0,>n})$ which is semistable of slope 0. By uniqueness of the Harder–Narasimhan filtration, it remains to prove that the Harder–Narasimhan slopes of $\mathcal{E}_+^n(V_{0,n})$ are all > 0.

Assume that 0 is a Harder–Narasimhan slope of $\mathcal{E}_+^n(V_{0,n})$, and let $\mathcal{H}$ be the first step of the Harder–Narasimhan filtration of $\mathcal{E}_+^n(V_{0,n})$, that is, the vector bundle $\mathcal{H}$ is semistable of slope 0 and there exists a surjective map

$$f : \mathcal{E}_+^n(V_{0,n}) \to \mathcal{H} \to 0.$$

By Remark 3.2.2, the vector bundle $\mathcal{H}$ and the map f are G_K-equivariant. Since $\mathcal{H}$ is semistable of slope 0, there exists a p-adic representation W of G_K such that $\mathcal{H} \xrightarrow{\sim} \mathcal{E}(W)$ by Theorem 3.4.2. Moreover, by Proposition 3.6.3, the surjection f implies that $\mathcal{H}$, and thus W, is de Rham with Hodge–Tate weights all in $\mathbf{Z} \setminus [1, n]$. Therefore, we have

$$\begin{aligned} f \in \operatorname{Hom}_{\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)^{\leq 0, > n}_{\mathrm{dR}}}(\mathcal{E}^n_+(V_{0,n}), \mathcal{H}) &\xrightarrow{\sim} \operatorname{Hom}_{\mathrm{Bun}_{X^{\mathrm{FF}}}(G_K)_{\mathrm{dR}}}(\mathcal{E}(V_{0,n}), \mathcal{H}) \\ &\xrightarrow{\sim} \operatorname{Hom}_{\mathrm{Rep}_{\mathbf{Q}_p}(G_K)}(V_{0,n}, W), \end{aligned}$$

where the first isomorphism follows from the adjunction from Proposition 5.2.1, the second from the equivalence from Theorem 3.4.2.

By Lemma 5.3.3 and Proposition 3.6.3, the group $\operatorname{Hom}_{\mathrm{Rep}_{\mathbf{Q}_p}(G_K)}(V_{0,n}, W)$ is trivial. In particular, the map f is trivial, which contradicts the assumption that 0 is a Harder–Narasimhan slope of $\mathcal{E}^n_+(V_{0,n})$. ∎

6 Bloch–Kato groups over perfectoid fields

6.1 The Bloch–Kato groups

We recall the definition of the Bloch–Kato groups [3, §3], and we define a filtration on the exponential Bloch–Kato groups induced by the filtration of $\mathbf{B}_{\mathrm{dR}}$.

Let V be a p-adic representation of G_K. For each finite extension K' of K, the *exponential*, *finite* and *geometric Bloch–Kato groups* are respectively defined by

$$\begin{aligned} \mathrm{H}^1_e(K', V) &= \operatorname{Ker}\left(\mathrm{H}^1(K', V) \to \mathrm{H}^1(K', \mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} V)\right), \\ \mathrm{H}^1_f(K', V) &= \operatorname{Ker}\left(\mathrm{H}^1(K', V) \to \mathrm{H}^1(K', \mathbf{B}_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V)\right), \\ \mathrm{H}^1_g(K', V) &= \operatorname{Ker}\left(\mathrm{H}^1(K', V) \to \mathrm{H}^1(K', \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)\right). \end{aligned}$$

Recall that the filtration of $\mathbf{B}_{\mathrm{dR}}$ induces a filtration on $\mathbf{B}_{\mathrm{e}}$. We define a filtration on the exponential Bloch–Kato groups as follows.

Definition 6.1.1. For each $n \in \mathbf{Z}$, we set

$$\operatorname{Fil}^n \mathrm{H}^1_e(K', V) = \begin{cases} \operatorname{Ker}\left(\mathrm{H}^1(K', V) \to \mathrm{H}^1(K', \operatorname{Fil}^n \mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} V)\right) & \text{if } n \leq 0, \\ 0 & \text{if } n \geq 0. \end{cases}$$

Let T be a G_K-stable lattice in V. The short exact sequence of topological G_K-modules

$$0 \to T \to V \to V/T \to 0$$

induces an exact sequence

$$\mathrm{H}^1(K', T) \xrightarrow{\alpha} \mathrm{H}^1(K', V) \xrightarrow{\beta} \mathrm{H}^1(K', V/T).$$

For $* \in \{e, f, g\}$, the Bloch–Kato subgroups of $\mathrm{H}^1(K', T)$ and $\mathrm{H}^1(K', V/T)$ are respectively defined by

$$\begin{aligned} \mathrm{H}^1_*(K', T) &= \alpha^{-1}\left(\mathrm{H}^1_*(K', V)\right), \\ \mathrm{H}^1_*(K', V/T) &= \beta\left(\mathrm{H}^1_*(K', V)\right). \end{aligned}$$

Moreover, the exponential Bloch–Kato groups are equipped with the induced filtrations

$$\begin{aligned}\mathrm{Fil}^n \mathrm{H}^1_e(K', T) &= \alpha^{-1}\left(\mathrm{Fil}^n \mathrm{H}^1_e(K', V)\right),\\ \mathrm{Fil}^n \mathrm{H}^1_e(K', V/T) &= \beta\left(\mathrm{Fil}^n \mathrm{H}^1_e(K', V)\right).\end{aligned}$$

Let L be an algebraic extension of K. Recall that there is a natural isomorphism

$$\mathrm{H}^1(L, V/T) \xrightarrow{\sim} \varinjlim_{\mathrm{res}, K'} \mathrm{H}^1(K', V/T),$$

where K' runs over all the finite extensions of K contained in L, and the transition morphisms are the restriction maps [32, I §2.2 Proposition 8]. For each $* \in \{e, f, g\}$, the groups $\mathrm{H}^1_*(K', V/T)$ and the filtration $\mathrm{Fil}^n \mathrm{H}^1_e(K', V/T)$ are compatible under the restriction maps, and the Bloch–Kato subgroups of $\mathrm{H}^1(L, V/T)$ are then defined by

$$\begin{aligned}\mathrm{H}^1_*(L, V/T) &= \varinjlim_{\mathrm{res}, K'} \mathrm{H}^1_*(K', V/T),\\ \mathrm{Fil}^n \mathrm{H}^1_e(L, V/T) &= \varinjlim_{\mathrm{res}, K'} \mathrm{Fil}^n \mathrm{H}^1_e(K', V/T),\end{aligned}$$

where K' runs over all the finite extensions of K contained in L, and the transition morphisms are the restriction maps.

6.2 Universal norms

Let V be a p-adic representation of G_K. Let T be a G_K-stable lattice in V. Let L be an algebraic extension of K.

For $i \in \mathbf{N}$, the *i-th Iwasawa cohomology group* $\mathrm{H}^i_{\mathrm{Iw}}(K, L, T)$ of the extension L/K with coefficients in T is defined by

$$\mathrm{H}^i_{\mathrm{Iw}}(K, L, T) = \varprojlim_{\mathrm{cores}, K'} \mathrm{H}^i(K', T),$$

where K' runs over all the finite extensions of K contained in L, and the transition morphisms are the corestriction maps.

For each $* \in \{e, f, g\}$, the Bloch–Kato groups $\mathrm{H}^1_*(K', T)$ are compatible under the corestriction maps. The modules of *$*$-universal norms* associated with T in the extension L/K are defined by

$$\mathrm{H}^1_{\mathrm{Iw},*}(K, L, T) = \varprojlim_{\mathrm{cores}, K'} \mathrm{H}^1_*(K', T),$$

where K' runs over all the finite extensions of K contained in L, and the transition morphisms are the corestriction maps.

Let $V^*(1) = \mathrm{Hom}_{\mathbf{Q}_p}(V, \mathbf{Q}_p(1))$ be the Tate dual representation of V, and let $T^*(1) = \mathrm{Hom}_{\mathbf{Z}_p}(T, \mathbf{Z}_p(1))$ be the G_K-stable lattice in $V^*(1)$ Tate dual of T. Recall that local Tate duality [32], for each finite extension K' of K,

$$\mathrm{H}^1(K', V^*(1)/T^*(1)) \times \mathrm{H}^1(K', T) \to \mathrm{H}^2(K', \mathbf{Q}_p(1)/\mathbf{Z}_p(1)) \cong \mathbf{Q}_p/\mathbf{Z}_p$$

induces a perfect pairing

$$\mathrm{H}^1(L, V^*(1)/T^*(1)) \times \mathrm{H}^1_{\mathrm{Iw}}(K, L, T) \to \mathbf{Q}_p/\mathbf{Z}_p.$$

Proposition 6.2.1 (Bloch–Kato)**.** *If V is de Rham, then, under local Tate duality,*

1. *the orthogonal complement of* $\mathrm{H}^1_e(L, V^*(1)/T^*(1))$ *is* $\mathrm{H}^1_{\mathrm{Iw},g}(K, L, T)$*,*
2. *the orthogonal complement of* $\mathrm{H}^1_f(L, V^*(1)/T^*(1))$ *is* $\mathrm{H}^1_{\mathrm{Iw},f}(K, L, T)$*,*
3. *the orthogonal complement of* $\mathrm{H}^1_g(L, V^*(1)/T^*(1))$ *is* $\mathrm{H}^1_{\mathrm{Iw},e}(K, L, T)$*.*

6.3 Comparison of the Bloch–Kato groups

Let V be a p-adic representation of G_K, and let T be a G_K-stable lattice in V. For each finite extension K' of K, let

$$\mathbf{D}_{\mathrm{cris},K'}(V) = \mathrm{H}^0(K', \mathbf{B}_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V).$$

Recall that $\mathbf{D}_{\mathrm{cris},K'}(V)$ is a finite dimensional K'_0-vector space equipped with a map φ semilinear with respect to the absolute Frobenius on K'_0 (see [17, §5.1]). For $i \in \mathbf{Z}$, we set the finite dimensional $\mathbf{Q}_p$-vector space

$$\mathbf{D}_{\mathrm{cris},K'}(V)^{\varphi=p^i} = \{x \in \mathbf{D}_{\mathrm{cris},K'}(V), \varphi(x) = p^i \cdot x\}.$$

Proposition 6.3.1. *Let $i \in \mathbf{Z}$. The dimension of the $\mathbf{Q}_p$-vector space $\mathbf{D}_{\mathrm{cris},K'}(V)^{\varphi=p^i}$ is bounded independently of K'.*

Proof. Let

$$\mathbf{D}_{\mathrm{pcris}}(V) = \varinjlim_{\mathrm{res},K'} \mathrm{H}^0(K', \mathbf{B}_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V),$$

where K' runs over all the finite extensions of K, and the transition morphisms are the restriction maps. Then $\mathbf{D}_{\mathrm{pcris}}(V)$ is a finite dimensional discrete (φ, G_K)-module over $\mathbf{Q}_p^{\mathrm{ur}}$, that is, $\mathbf{D}_{\mathrm{pcris}}(V)$ is a finite dimensional $\mathbf{Q}_p^{\mathrm{ur}}$-vector space equipped with a map φ semilinear with respect to the absolute Frobenius on $\mathbf{Q}_p^{\mathrm{ur}}$ and a discrete action of G_K commuting with φ (see [17, §5.6]). We set

$$\widehat{\mathbf{D}}_{\mathrm{pcris}}(V) = \hat{\mathbf{Q}}_p^{\mathrm{ur}} \otimes_{\mathbf{Q}_p^{\mathrm{ur}}} \mathbf{D}_{\mathrm{pcris}}(V).$$

Then (see [17, Remarque 4.4.10]), $\widehat{\mathbf{D}}_{\mathrm{pcris}}(V)$ is a finite dimensional (φ, G_K)-module over $\hat{\mathbf{Q}}_p^{\mathrm{ur}}$, and for each finite extension K' of K, we have

$$\mathbf{D}_{\mathrm{cris},K'}(V) = \mathbf{D}_{\mathrm{pcris}}(V)^{G_{K'}} = \widehat{\mathbf{D}}_{\mathrm{pcris}}(V)^{G_{K'}}.$$

Hence, since the action of G_K and φ commute, we have

$$\mathbf{D}_{\mathrm{cris},K'}(V)^{\varphi=p^i} = (\widehat{\mathbf{D}}_{\mathrm{pcris}}(V)^{\varphi=p^i})^{G_{K'}}.$$

By the Dieudonné–Manin theorem [12, IV §4], there is an isomorphism of φ-modules over $\hat{\mathbf{Q}}_p^{\mathrm{ur}}$

$$\widehat{\mathbf{D}}_{\mathrm{pcris}}(V) \xrightarrow{\sim} \bigoplus_{v \in \mathbf{Q}} E(v)^{\oplus m_v},$$

where $E(v)$ runs over the simple objects in the category of the φ-modules over $\hat{\mathbf{Q}}_p^{\mathrm{ur}}$, that is, if $v = s/r$ with $s, r \in \mathbf{N}$, $r > 0$, and $(r, s) = 1$, then $E(v) = \hat{\mathbf{Q}}_p^{\mathrm{ur}} \otimes_{\mathbf{Q}_p} \mathbf{Q}_p[T]/(T^r - p^s)$ on which φ acts by multiplication by T semilinear with respect to the absolute Frobenius on $\hat{\mathbf{Q}}_p^{\mathrm{ur}}$. We then have

$$\widehat{\mathbf{D}}_{\mathrm{pcris}}(V)^{\varphi=p^i} \xrightarrow{\sim} (E(i)^{\varphi=p^i})^{\oplus m_i},$$

and $E(i)^{\varphi=p^i}$ is a finite dimensional $\mathbf{Q}_p$-vector space [12, IV §2 and §3]. ∎

Proposition 6.3.2 (Bloch–Kato). *Let K' be a finite extension of K. If V is de Rham, then we have*

$$\begin{aligned}\dim_{\mathbf{Q}_p} \mathrm{H}^1_f(K', V)/\,\mathrm{H}^1_e(K', V) &= \dim_{\mathbf{Q}_p} \mathbf{D}_{\mathrm{cris},K'}(V)^{\varphi=1},\\ \dim_{\mathbf{Q}_p} \mathrm{H}^1_g(K', V)/\,\mathrm{H}^1_f(K', V) &= \dim_{\mathbf{Q}_p} \mathbf{D}_{\mathrm{cris},K'}(V)^{\varphi=p^{-1}}.\end{aligned}$$

Proof. Bloch and Kato [3, Corollary 3.8.4] have proved that there exists an isomorphism of $\mathbf{Q}_p$-vector spaces

$$\mathrm{H}^1_f(K', V)/\,\mathrm{H}^1_e(K', V) \xrightarrow{\sim} \mathbf{D}_{\mathrm{cris},K'}(V)/(1-\varphi)\,\mathbf{D}_{\mathrm{cris},K'}(V),$$

which implies the first statement. By the duality from Proposition 6.2.1, the first statement yields

$$\begin{aligned}\dim_{\mathbf{Q}_p} \mathrm{H}^1_g(K', V)/\,\mathrm{H}^1_f(K', V) &= \dim_{\mathbf{Q}_p} \mathrm{H}^1_f(K', V^*(1))/\,\mathrm{H}^1_e(K', V^*(1))\\ &= \dim_{\mathbf{Q}_p} \mathbf{D}_{\mathrm{cris},K'}(V^*(1))^{\varphi=1},\end{aligned}$$

and the duality of φ-modules [17, §5.1]

$$\mathbf{D}_{\mathrm{cris},K'}(V) \otimes_{K'_0} \mathbf{D}_{\mathrm{cris},K'}(V^*(1)) \xrightarrow{\sim} \mathbf{D}_{\mathrm{cris},K'}(\mathbf{Q}_p(1)),$$

implies the equality

$$\dim_{\mathbf{Q}_p} \mathbf{D}_{\mathrm{cris},K'}(V^*(1))^{\varphi=1} = \dim_{\mathbf{Q}_p} \mathbf{D}_{\mathrm{cris},K'}(V)^{\varphi=p^{-1}}.$$

∎

Proposition 6.3.3. *Let L be an algebraic extension of K. If V is de Rham, then the Pontryagin dual of the quotient*

$$\mathrm{H}^1_g(L, V/T)/\,\mathrm{H}^1_e(L, V/T)$$

is a free $\mathbf{Z}_p$-module of finite rank bounded independently of L.

Proof. By Proposition 6.2.1, the Pontryagin dual of the discrete $\mathbf{Z}_p$-module

$$\mathrm{H}^1_g(L, V/T)/\,\mathrm{H}^1_e(L, V/T)$$

is the compact $\mathbf{Z}_p$-module

$$\mathrm{H}^1_{\mathrm{Iw},g}(K, L, T^*(1))/\,\mathrm{H}^1_{\mathrm{Iw},e}(K, L, T^*(1)).$$

By definition, there exists an injective map

$$0 \to \mathrm{H}^1_{\mathrm{Iw},g}(K,L,T^*(1))/\,\mathrm{H}^1_{\mathrm{Iw},e}(K,L,T^*(1)) \to \varprojlim_{\mathrm{cores},K'} \mathrm{H}^1_g(K',V^*(1))/\,\mathrm{H}^1_e(K',V^*(1)), \tag{6.3.1}$$

where K' runs over all the finite extensions of K contained in L, and the transition morphisms are the corestriction maps.

By Proposition 6.3.1 and Proposition 6.3.2, the dimension of the $\mathbf{Q}_p$-vector space $\mathrm{H}^1_g(K',V^*(1))/\,\mathrm{H}^1_e(K',V^*(1))$ is bounded independently of K'. Therefore, the $\mathbf{Q}_p$-vector space $\varprojlim \mathrm{H}^1_g(K',V^*(1))/\,\mathrm{H}^1_e(K',V^*(1))$ is finite dimensional, and we conclude using the map (6.3.1). ∎

6.4 Universal extensions and groups of points

We recall the definition and properties of universal objects in $\mathcal{C}(G_K)$ and of the groups of points both associated with a p-adic representation by Fontaine [18, §8].

Let V be a p-adic representation of G_K. The tangent space t_V associated with V is the K-vector space

$$t_V = ((\mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V)^{G_K},$$

which is equipped with the filtration by K-vector subspaces

$$\mathrm{Fil}^n t_V = \begin{cases} ((\mathrm{Fil}^n \mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V)^{G_K} & \text{if } n \le 0 \\ 0 & \text{if } n > 0. \end{cases}$$

If V is de Rham, then they are isomorphisms

$$\mathbf{D}_{\mathrm{dR}}(V)/\,\mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V) \xrightarrow{\sim} t_V,$$

and, for each $n \in \mathbf{N}$,

$$\mathrm{Fil}^{-n} \mathbf{D}_{\mathrm{dR}}(V)/\,\mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V) \xrightarrow{\sim} \mathrm{Fil}^{-n} t_V.$$

We set

$$t_V(\bar{\mathbf{Q}}_p) = ((\mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V)_\delta,$$
$$\mathrm{Fil}^{-n} t_V(\bar{\mathbf{Q}}_p) = ((\mathrm{Fil}^{-n} \mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V)_\delta.$$

Let $\hat{t}_V(\bar{\mathbf{Q}}_p)$ be the topological closure of the image of $t_V(\bar{\mathbf{Q}}_p)$ in $(\mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V$, and, for each $n \in \mathbf{N}$, let $\mathrm{Fil}^{-n} \hat{t}_V(\bar{\mathbf{Q}}_p)$ be the topological closure of the image of $\mathrm{Fil}^{-n} t_V(\bar{\mathbf{Q}}_p)$ in $(\mathrm{Fil}^{-n} \mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V$.

Let $t_V(\mathbf{B}^+_{\mathrm{dR}}) = t_V \otimes_K \mathbf{B}^+_{\mathrm{dR}}$, and, for each $n \in \mathbf{N}$, let $\mathrm{Fil}^{-n} t_V(\mathbf{B}^+_{\mathrm{dR}}) = \mathrm{Fil}^{-n} t_V \otimes_K \mathbf{B}^+_{\mathrm{dR}}$. Note that there are natural morphisms of $\mathbf{B}^+_{\mathrm{dR}}$-modules by extension of scalars

$$t_V(\mathbf{B}^+_{\mathrm{dR}}) \to (\mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V$$

and, for each $n \in \mathbf{N}$,

$$\mathrm{Fil}^{-n} t_V(\mathbf{B}^+_{\mathrm{dR}}) \to (\mathrm{Fil}^{-n} \mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V.$$

A *trivial torsion* $\mathbf{B}_{\mathrm{dR}}^{+}$*-representation of* G_K (respectively a *trivial* $\mathbf{B}_n$*-representation of* G_K) is a torsion $\mathbf{B}_{\mathrm{dR}}^{+}$-representation of G_K isomorphic to $\bigoplus_{i\in\mathbf{N}} \mathbf{B}_i^{\oplus m_i}$ (respectively $\bigoplus_{i\in[1,n]} \mathbf{B}_i^{\oplus m_i}$), for some integers m_i.

Proposition 6.4.1. *The modules associated with the tangent space of V satisfy the following properties.*

1. (a) *There is an isomorphism of discrete* G_K*-modules*

$$t_V(\bar{\mathbf{Q}}_p) \xrightarrow{\sim} t_V \otimes_K \bar{\mathbf{Q}}_p$$

   (b) *The module* $\hat{t}_V(\bar{\mathbf{Q}}_p)$ *is the maximal trivial torsion* $\mathbf{B}_{\mathrm{dR}}^{+}$*-subrepresentation of* $(\mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^{+}) \otimes_{\mathbf{Q}_p} V$.

   (c) *There is an isomorphism of torsion* $\mathbf{B}_{\mathrm{dR}}^{+}$*-representations of* G_K

$$\hat{t}_V(\bar{\mathbf{Q}}_p) \xrightarrow{\sim} \mathrm{Im}\left(t_V(\mathbf{B}_{\mathrm{dR}}^{+}) \to (\mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^{+}) \otimes_{\mathbf{Q}_p} V\right).$$

   (d) *If V is de Rham, then there exists an isomorphism of* $\mathbf{B}_{\mathrm{dR}}^{+}$*-representations of* G_K

$$\hat{t}_V(\bar{\mathbf{Q}}_p) \xrightarrow{\sim} \bigoplus_{i\in\mathbf{N}} \mathbf{B}_i^{\oplus m_i(V)},$$

   where $m_i(V)$ *is the multiplicity of i as a Hodge–Tate weight of V.*

2. (a) *There is an isomorphism of discrete* G_K*-modules*

$$\mathrm{Fil}^{-n}\, t_V(\bar{\mathbf{Q}}_p) \xrightarrow{\sim} \mathrm{Fil}^{-n}\, t_V \otimes_K \bar{\mathbf{Q}}_p$$

   (b) *The module* $\mathrm{Fil}^{-n}\, \hat{t}_V(\bar{\mathbf{Q}}_p)$ *is the maximal trivial torsion* $\mathbf{B}_n$*-subrepresentation of* $(\mathrm{Fil}^{-n}\, \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^{+}) \otimes_{\mathbf{Q}_p} V$.

   (c) *There is an isomorphism of torsion* $\mathbf{B}_{\mathrm{dR}}^{+}$*-representations of* G_K

$$\mathrm{Fil}^{-n}\, \hat{t}_V(\bar{\mathbf{Q}}_p) \xrightarrow{\sim} \mathrm{Im}\left(\mathrm{Fil}^{-n}\, t_V(\mathbf{B}_{\mathrm{dR}}^{+}) \to (\mathrm{Fil}^{-n}\, \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^{+}) \otimes_{\mathbf{Q}_p} V\right).$$

   (d) *If V is de Rham, then there exists an isomorphism of* $\mathbf{B}_{\mathrm{dR}}^{+}$*-representations of* G_K

$$\mathrm{Fil}^{-n}\, \hat{t}_V(\bar{\mathbf{Q}}_p) \xrightarrow{\sim} \bigoplus_{i\in[1,n]} \mathbf{B}_i^{\oplus m_i(V)},$$

   where $m_i(V)$ *is the multiplicity of i as a Hodge–Tate weight of V.*

Proof. The first three points in both cases are due to Fontaine [18, Proposition 8.1]. For last points under the assumption that V is de Rham, the statement for $\hat{t}_V(\bar{\mathbf{Q}}_p)$ is [29, Corollary 3.3.4], while the statement for $\mathrm{Fil}^n\, \hat{t}_V(\bar{\mathbf{Q}}_p)$ is proved similarly. ∎

We fix an integer $n \geq 1$. We set $E_{\mathrm{e}}(V) = \mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} V$, and $E_n(V) = \mathrm{Fil}^{-n}\,\mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} V$. The tensor product of V with the fundamental exact sequences (2.2.2) and (2.2.3) yields a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] & E_e(V) \arrow[r] & (\mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^+) \otimes_{\mathbf{Q}_p} V \arrow[r] & 0 \\ 0 \arrow[r] & V \arrow[r] \arrow[u, equal] & E_n(V) \arrow[r] \arrow[u] & (\mathrm{Fil}^{-n}\,\mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^+) \otimes_{\mathbf{Q}_p} V \arrow[r] \arrow[u] & 0. \end{tikzcd} \tag{6.4.1}$$

Let $E_+(V)$ be the reciprocal image of $\hat{t}_V(\bar{\mathbf{Q}}_p)$ in $E_{\mathrm{e}}(V)$, and let $E_+^n(V)$ be the reciprocal image of $\mathrm{Fil}^{-n}\,\hat{t}_V(\bar{\mathbf{Q}}_p)$ in $E_n(V)$. The diagram (6.4.1) induces a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] & E_+(V) \arrow[r] & \hat{t}_V(\bar{\mathbf{Q}}_p) \arrow[r] & 0 \\ 0 \arrow[r] & V \arrow[r] \arrow[u, equal] & E_+^n(V) \arrow[r] \arrow[u] & \mathrm{Fil}^{-n}\,\hat{t}_V(\bar{\mathbf{Q}}_p) \arrow[r] \arrow[u] & 0. \end{tikzcd} \tag{6.4.2}$$

Proposition 6.4.2 (Fontaine). *The topological G_K-modules $E_+(V)$ and $E_+^n(V)$ are almost $\mathbf{C}_p$-representations of G_K. Moreover, they satisfy the following universal properties.*

1. *The almost $\mathbf{C}_p$-representation $E_+(V)$ is the universal extension of V by a trivial torsion $\mathbf{B}_{\mathrm{dR}}^+$-representation in $\mathcal{B}(G_K)$.*

2. *The almost $\mathbf{C}_p$-representation $E_+^n(V)$ is the universal extension of V by a trivial $\mathbf{B}_n$-representation in $\mathcal{B}(G_K)$.*

Lemma 6.4.3. *Assume that V is de Rham.*

1. *If the Hodge–Tate weights of V are all ≤ 0, then*

$$V = E_+^n(V) = E_+(V).$$

2. *If the Hodge–Tate weights of V are all $\leq n$, then*

$$E_+^n(V) = E_+(V).$$

Proof. Both statements follow from Proposition 6.4.1. ∎

Proposition 6.4.4. *If V is de Rham, then there exists an isomorphism of short exact sequences of almost $\mathbf{C}_p$-representations of G_K*

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] \arrow[d, "\wr"] & E_+^n(V) \arrow[r] \arrow[d, "\wr"] & \mathrm{Fil}^{-n}\,\hat{t}_V(\bar{\mathbf{Q}}_p) \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & \mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{E}(V)) \arrow[r] & \mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{E}_+^n(V)) \arrow[r] & \mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{F}_+^n(V)) \arrow[r] & 0. \end{tikzcd}$$

Proof. The statement is proved similarly to [29, Proposition 3.3.1 and Lemma 3.3.2]. By Lemma 5.3.1, the Harder–Narasimhan slopes of the sheaves

$$0 \to \mathcal{E}(V) \to \mathcal{E}^n_+(V) \to \mathcal{F}^n_+(V) \to 0$$

are all ≥ 0. Thus, by Proposition 3.2.1, there exists a commutative diagram of topological G_K-modules

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r] & E_{\mathrm{e}}(V) \arrow[r] & (\mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{E}(V)) \arrow[r] \arrow[u, "\wr"] & \mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{E}^n_+(V)) \arrow[r] \arrow[u] & \mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{F}^n_+(V)) \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

where the bottom rows is a short exact sequence of almost $\mathbf{C}_p$-representations of G_K by Theorem 3.4.2. Since V is de Rham, there are isomorphisms of torsion $\mathbf{B}^+_{\mathrm{dR}}$-representations

$$\begin{aligned}
\mathrm{Fil}^{-n}\,\hat{t}_V(\bar{\mathbf{Q}}_p) &\xrightarrow{\sim} \mathrm{Im}\left(\mathrm{Fil}^{-n}\,t_V(\mathbf{B}^+_{\mathrm{dR}}) \to (\mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V\right) \\
&\xrightarrow{\sim} \mathrm{Im}\left(\mathbf{B}^+_{\mathrm{dR}} \otimes_K (\mathrm{Fil}^{-n}\,\mathbf{D}_{\mathrm{dR}}(V)/\,\mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}(V)) \to (\mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V\right) \\
&\xrightarrow{\sim} \mathrm{Im}\left(\frac{\mathbf{B}^+_{\mathrm{dR}} \otimes_K \mathrm{Fil}^{-n}\,\mathbf{D}_{\mathrm{dR}}(V)}{\mathbf{B}^+_{\mathrm{dR}} \otimes_K \mathrm{Fil}^0\,\mathbf{D}_{\mathrm{dR}}(V)} \to \frac{\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V}{\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V}\right) \\
&\xrightarrow{\sim} \frac{\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V + \mathbf{B}^+_{\mathrm{dR}} \otimes_K \mathrm{Fil}^{-n}\,\mathbf{D}_{\mathrm{dR}}(V)}{\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V} \\
&\xrightarrow{\sim} \mathrm{H}^0(X^{\mathrm{FF}}, \mathcal{F}^n_+(V)),
\end{aligned}$$

where the first isomorphism is from Proposition 6.4.1 and the last one from Theorem 3.4.2. ∎

The combination of Proposition 6.4.4 and Proposition 5.3.4 yields the following.

Corollary 6.4.5. *If V is de Rham, then there is a natural isomorphism of p-adic representation of G_K*

$$E^n_+(V)^0 \xrightarrow{\sim} V^{\leq 0, > n}.$$

Let T be a G_K-stable lattice in V. We set $E_+(V/T) = E_+(V)/T$ and $E^n_+(V/T) = E^n_+(V)/T$, and thus the diagram (6.4.2) induces a commutative diagram of topological G_K-modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & V/T \arrow[r] & E_+(V/T) \arrow[r] & \hat{t}_V(\bar{\mathbf{Q}}_p) \arrow[r] & 0 \\
0 \arrow[r] & V/T \arrow[r] \arrow[u, equal] & E^n_+(V/T) \arrow[r] \arrow[u] & \mathrm{Fil}^{-n}\,\hat{t}_V(\bar{\mathbf{Q}}_p) \arrow[r] \arrow[u] & 0.
\end{tikzcd} \tag{6.4.3}$$

We set $E_\delta(V/T) = (E_+(V/T))_\delta$ and $E^n_\delta(V/T) = (E^n_+(V/T))_\delta$, and thus, by Lemma 4.2.2, the diagram (6.4.3) induces a commutative diagram of discrete

G_K-modules with exact rows

$$\begin{tikzcd} 0 \arrow[r] & V/T \arrow[r] & E_\delta(V/T) \arrow[r] & t_V(\bar{\mathbf{Q}}_p) \arrow[r] & 0 \\ 0 \arrow[r] & V/T \arrow[r] \arrow[u, equal] & E_\delta^n(V/T) \arrow[r] \arrow[u] & \mathrm{Fil}^{-n}\, t_V(\bar{\mathbf{Q}}_p) \arrow[r] \arrow[u] & 0 \end{tikzcd} \tag{6.4.4}$$

Remark 6.4.6. Fontaine has defined the *group of points* $E_{\mathrm{disc}}(V/T)$ associated with V/T as the image of $E_\delta(V/T)$ in $E_+(V/T)$. As in the article [29], we use different notation to highlight the different topologies: $E_\delta(V/T)$ is a discrete G_K-module, while $E_{\mathrm{disc}}(V/T)$ is endowed with the subspace topology from $E_+(V/T)$.

Proposition 6.4.7. *Let L be an algebraic extension of K. The commutative diagram of discrete G_K-modules* (6.4.4) *induces a commutative diagram whose rows are exact*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^1_e(L,V/T) \arrow[r] & \mathrm{H}^1(L,V/T) \arrow[r] & \mathrm{H}^1(L,E_\delta(V/T)) \arrow[r] & 0 \\ 0 \arrow[r] & \mathrm{Fil}^{-n}\,\mathrm{H}^1_e(L,V/T) \arrow[r] \arrow[u] & \mathrm{H}^1(L,V/T) \arrow[r] \arrow[u, equal] & \mathrm{H}^1(L,E^n_\delta(V/T)) \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

Proof. The statement for $E_\delta(V/T)$ is [29, Proposition 3.2.1], while the statement for $E^n_\delta(V/T)$ is proved similarly. If K' is a finite extension of K, then the cohomology of K' of the diagram

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] & E_n(V) \arrow[r] & (\mathrm{Fil}^{-n}\,\mathbf{B}_{\mathrm{dR}}/\mathbf{B}^+_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V \arrow[r] & 0 \\ 0 \arrow[r] & V \arrow[r] \arrow[u, equal] & E^n_+(V) \arrow[r] \arrow[u] & \mathrm{Fil}^{-n}\,\hat{t}_V(\bar{\mathbf{Q}}_p) \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

induces the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & V^{G_{K'}} \arrow[r] & \mathrm{Fil}^{-n}\,\mathbf{D}_{\mathrm{cris},K'}(V)^{\varphi=1} \arrow[r] & \mathrm{Fil}^{-n}\,t_V(K') \arrow[r] & \mathrm{H}^1(K',V) \\ 0 \arrow[r] & V^{G_{K'}} \arrow[r] \arrow[u, equal] & E^n_+(V)^{G_{K'}} \arrow[r] \arrow[u, "\wr"] & \mathrm{Fil}^{-n}\,t_V(K') \arrow[r] \arrow[u, equal] & \mathrm{H}^1(K',V). \arrow[u, equal] \end{tikzcd}$$

Hence, by definition of $\mathrm{Fil}^{-n}\,\mathrm{H}^1_e(K',V)$, there is an exact sequence

$$0 \to V^{G_{K'}} \to E^n_+(V)^{G_{K'}} \to \mathrm{Fil}^{-n}\,t_V(K') \to \mathrm{Fil}^{-n}\,\mathrm{H}^1_e(K',V) \to 0. \tag{6.4.5}$$

The commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] \arrow[d] & E^n_+(V) \arrow[r] \arrow[d] & \mathrm{Fil}^{-n}\,\hat{t}_V(\bar{\mathbf{Q}}_p) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & V/T \arrow[r] & E^n_+(V/T) \arrow[r] & \mathrm{Fil}^{-n}\,\hat{t}_V(\bar{\mathbf{Q}}_p) \arrow[r] & 0 \end{tikzcd}$$

induces the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V^{G_{K'}} & \longrightarrow & E^n_+(V)^{G_{K'}} & \longrightarrow & \mathrm{Fil}^{-n}\, t_V(K') & \longrightarrow & \mathrm{H}^1(K',V) \\ & & \downarrow & & \downarrow & & \| & & \downarrow \\ 0 & \longrightarrow & (V/T)^{G_{K'}} & \longrightarrow & E^n_+(V/T)^{G_{K'}} & \longrightarrow & \mathrm{Fil}^{-n}\, t_V(K') & \longrightarrow & \mathrm{H}^1(K',V/T). \end{array} \tag{6.4.6}$$

Hence, by definition of $\mathrm{Fil}^{-n}\,\mathrm{H}^1_e(K',V/T)$, the exact sequence (6.4.5), and the commutativity of the diagram (6.4.6), there is an exact sequence

$$0 \to (V/T)^{G_{K'}} \to E^n_+(V/T)^{G_{K'}} \to \mathrm{Fil}^{-n}\, t_V(K') \to \mathrm{Fil}^{-n}\,\mathrm{H}^1_e(K',V/T) \to 0. \tag{6.4.7}$$

Therefore, by the exact sequence (6.4.7), the short exact sequence of discrete G_K-modules

$$0 \to V/T \to E^n_\delta(V/T) \to \mathrm{Fil}^{-n}\, t_V(\bar{\mathbf{Q}}_p) \to 0$$

induces a short exact sequence

$$0 \to \mathrm{Fil}^{-n}\,\mathrm{H}^1_e(K',V/T) \to \mathrm{H}^1(K',V/T) \to \mathrm{H}^1(K',E^n_\delta(V/T)) \to 0. \tag{6.4.8}$$

By definition of $\mathrm{Fil}^{-n}\,\mathrm{H}^1_e(L,V/T)$ and by continuity of Galois cohomology with coefficients in discrete modules [32, I §2.2 Proposition 8], the limit of the short exact sequences (6.4.8) over the finite extensions K' of K contained in L with transition morphisms the restriction maps yields

$$0 \to \mathrm{Fil}^{-n}\,\mathrm{H}^1_e(L,V/T) \to \mathrm{H}^1(L,V/T) \to \mathrm{H}^1(L,E^n_\delta(V/T)) \to 0.$$

∎

The combination of Proposition 6.4.7 and Lemma 6.4.3 yields the following.

Corollary 6.4.8. *Let L be an algebraic extension of K. Assume that V is de Rham.*

1. *If the Hodge–Tate weights of V are all ≤ 0, then*

$$\mathrm{H}^1_e(L,V/T) = 0.$$

2. *If the Hodge–Tate weights of V are all $\leq n$, then*

$$\mathrm{Fil}^{-n}\,\mathrm{H}^1_e(L,V/T) = \mathrm{H}^1_e(L,V/T).$$

6.5 Bloch–Kato groups and Galois theory of $\mathbf{B}^+_{\mathrm{dR}}$

Let V be a p-adic representation of G_K. Let L be an algebraic extension of K.

If V is de Rham and if $n \geq 1$ is an integer, then we denote by $T^{\leq 0,>n}$ the image of T in $V^{\leq 0,>n}$, and the quotient map $V/T \to V^{\leq 0,>n}/T^{\leq 0,>n}$ induces a morphism

$$\pi_{0,n} : \mathrm{H}^1(L,V/T) \to \mathrm{H}^1(L,V^{\leq 0,>n}/T^{\leq 0,>n}).$$

Note that if the Hodge–Tate weights of V are all $\leq n$, then the representation $V^{\leq 0,>n}$ is the maximal quotient representation of V whose Hodge–Tate weights are all ≤ 0, and we then simply denote the representation $V^{\leq 0,>n}$ by $V^{\leq 0}$, the lattice $T^{\leq 0,>n}$ by $T^{\leq 0}$, and the map $\pi_{0,n}$ by

$$\pi_0 : \mathrm{H}^1(L, V/T) \to \mathrm{H}^1(L, V^{\leq 0}/T^{\leq 0}).$$

Theorem 6.5.1. *Let $n \geq 1$ be an integer. If V is de Rham and if $\hat{L}$ is a perfectoid field such that L is dense in $\mathbf{B}_n^{G_L}$, then the map $\pi_{0,n}$ induces an isomorphism*

$$\mathrm{H}^1(L, V/T)/\operatorname{Fil}^{-n}\mathrm{H}^1_e(L, V/T) \xrightarrow{\sim} \mathrm{H}^1(L, V^{\leq 0,>n}/T^{\leq 0,>n}).$$

Proof. By Proposition 6.4.7, there is an isomorphism

$$\mathrm{H}^1(L, V/T)/\operatorname{Fil}^{-n}\mathrm{H}^1_e(L, V/T) \xrightarrow{\sim} \mathrm{H}^1(L, E^n_\delta(V/T)).$$

Since L is dense in $\mathbf{B}_n^{G_L}$, by Proposition 6.4.1, $(\operatorname{Fil}^{-n} t_V(\bar{\mathbf{Q}}_p))^{G_L}$ is dense in $(\operatorname{Fil}^{-n} \hat{t}_V(\bar{\mathbf{Q}}_p))^{G_L}$. Therefore, by Corollary 4.2.5 and Corollary 6.4.5, there are isomorphisms

$$\mathrm{H}^1(L, E^n_\delta(V/T)) \xrightarrow{\sim} \mathrm{H}^1(L, E^n_+(V/T)) \xrightarrow{\sim} \mathrm{H}^1(L, V^{\leq 0,>n}/T^{\leq 0,>n}).$$

∎

Corollary 6.5.2. *Let $n \geq 1$ be an integer. Assume that V is de Rham and that $\hat{L}$ is a perfectoid field such that L is dense in $\mathbf{B}_n^{G_L}$.*

1. *If the quotient representation $V^{\leq 0,>n}$ is trivial, then*

$$\mathrm{H}^1_e(L, V/T) = \mathrm{H}^1(L, V/T).$$

2. *If the Hodge–Tate weights of V are all $\leq n$, then the map π_0 induces an isomorphism*

$$\mathrm{H}^1(L, V/T)/\mathrm{H}^1_e(L, V/T) \xrightarrow{\sim} \mathrm{H}^1(L, V^{\leq 0}/T^{\leq 0}).$$

Proof. The first statement follows immediately from Theorem 6.5.1. The second statement follows from Theorem 6.5.1 and Corollary 6.4.8. ∎

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